THE PROBLEM REGARDING (RE)PLACEMENT OF A KING: AN INTUITION PUMP APPROACH

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In many school-based probability calculations, replacement of items that have been drawn in an event that is part of an outcome returns the sample space for the next event to the status quo. Consequently, replacement of the items has no impact on the probability of the upcoming event. The two events are independent. Intuitively, it is likewise reasoned that if a selected item for a first event is not replaced into the sample space, there will be a resulting change in the probability of the next event compared to when the items were replaced. The intuition pump in this article explores the validity of this intuitive assumption.

In math classes around the world, students are taught about independent and dependent events through a discussion of items, hereafter referred to as elements, from a sample space being replaced or not replaced in multiple-event outcomes. Consider the following pair of questions that are typical of the kind that might be presented to the students during this lesson:

1. A bag contains 5 red marbles and 5 blue marbles. A marble is drawn from the bag and then is replaced. A second marble is then drawn. What is the probability that the second marble is blue?
2. A bag contains 5 red marbles and 5 blue marbles. A marble is drawn from the bag and then not replaced. A second marble is then drawn. What is the probability that the second marble is blue?

Often in classrooms, the fundamental difference between these two questions is framed around whether the first marble is "replaced" or "not replaced." The notes provided on the board (which are diligently copied by the students into their notebooks) explain that the events are independent in question 1 because the first marble is replaced, and that the events in question 2 are dependent because the first marble is not replaced. The lesson likely then goes on to additional problems comprised of multiple-events outcomes involving replacement and not replacement, with the possibility of alternative representations being used (such as the Bayesian formula, Venn diagrams, or tree diagrams). In no time, the lesson is learned—"replacement = independent, not replacement = dependent"—an easy "trick" for remembering what to do when calculating probabilities of multiple-event outcomes. We contend, however, that while knowing whether there is replacement or not replacement is necessary for determining whether two events are independent or dependent, it is not sufficient information for reaching the conclusion that the two events are, in fact, independent or dependent.

Chernoff and Chernoff (2015) wrote about a similar suite of two problems, this time involving the drawing of cards from a standard deck rather than marbles from a bag. The first problem, A, reads as follows: "A card is drawn from the deck; the card is replaced; then a second card is drawn from the deck. What is the probability that the second card drawn is a king" (p. 30). It is a typical question in which the first event occurs, the element selected is returned to the sample space, and the second event occurs. The events in this "with replacement" problem are clearly (and calculatedly) independent. The second problem, B, is a slight variation of the first:

> A card is drawn from the deck; the card is not replaced, but, rather, is placed face down beside the deck; then a second card is drawn from the deck. What is the probability that the second card drawn is a king. (p. 30)

In this problem, the phrase, "not replaced," triggers the school-trained intuitive response or secondary intuition (Fischbein, 1975)—the two events are obviously dependent. But there is a problem: these two card problems have the same solution—the "dependent problem" probability equals the "independent problem" probability.

SOLUTIONS TO THE PROBLEMS

Before we take the reader into an explanation of the method we will be using to make sense of equivalency of these probabilities (and seemingly then, of independence and dependence), we first do the probability calculations for the two king problems, (A and B). In problem A, because the first card

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is replaced and because we are only concerned about the probability of the second card being a king, we can conclude that the sample space contains 52 outcomes (the 52 cards) with only four of those outcomes being kings (there are 13 ranks in the deck of cards, one of which is kings), so the P(2nd card is a king) = 4/52.

Intuitively, problem B seems just as easy at first. One card has been removed and not replaced, so we only have 51 cards (so a sample space of 51 outcomes) from which to choose the second card (hopefully a king). Next, we need to know how many kings are within that sample space, and it is at that point that we realize that we do not know what the first card is—it may be a king as well, but it may not. To calculate the actual probability, therefore, we need to determine the probability of the two possibilities or cases:

- The probability that the first card is a king and the second card is a king, which can be calculated as P(1st card is a king) x P(2nd card is a king) = 4/52 x 3/51; and
- The probability that the first card is not a king and the second card is a king, which can be calculated as 48/52 x 4/51.

The sum of these two probabilities, then, is the probability that the second card drawn is a king when the first card is left face down on the table beside the deck of cards (not replaced and not seen), or 4/52 (see Chernoff & Chernoff, 2015, p. 31 for full calculation details), which is exactly the same probability as if the first card had been replaced. Put another way, the dependent and independent variations of the general problem (what is the probability that the second card drawn from a standard deck of 52 cards is a king) have the same probability! At this point, it is typical for people's intuitions to be crying "foul!" and that is where our choice to utilize an intuition pump comes in.

METHOD

In this theoretical study, we have chosen to turn to philosophical research for an appropriate tool (e.g., Dowding, 2016). In particular, we were looking for a tool that would allow us to engage with our intuitions related to the "replaced equals not replaced problem" (Chernoff & Chernoff, 2015, p. 30) that is encountered in some contexts requiring consideration of the dependence or independence of events within an outcome. The tool that we ultimately chose was the intuition pump. An intuition pump is not a formal argument; rather it is a story, or series of stories, that are designed to prime and clarify our intuitions. Daniel Dennett, who coined the term, states: "intuition pumps are particularly valuable when there's confusion about just what the right questions are and what … matters to answer the question" (Big Think, 2014, 0:42). Further, Dorbolo (2013) explains that intuition pumps can be educational in that they "transform thoughts, convictions, judgments, decisions, and actions using intuitions elicited via reconceptualization" (p. 1644). Our problem is that two related problems, one that our intuition tells us involves independent events and the other that our intuition tells us involves dependent events, have the same answer. Thus, our problem presents the kind of confusion that Dennett is speaking of. In addition, we also have the goal of educating ourselves and others about what matters when determining the state of dependence within multiple-event probability problems. We contend that the intuition pump is a perfect fit for our research.

Our next task was to determine how to design and implement an intuition pump. Dennett (2013) tells of how his colleague and co-author of *The Mind's I* (Hofstadter & Dennett,1982), Douglas Hofstadter, suggested that one "consider the intuition pump to be a tool with many settings, and 'turn all the knobs' to see if the same intuitions still get pumped when you consider variations" (Dennett, 2013, p.7). Therefore, to carry out a valid intuition pump, we needed to determine what knob(s) existed and were of significance for our question. For the two king problems, there are numerous knobs that could be turned, such as the knob that controls who sees the card (no one, a bystander but not the person drawing the card, the person drawing the card but not a bystander, or everyone), or the knob that controls how people see the card (are they colour blind, can they distinguish between different hues, or can they see enough of the face of the card to know what its rank is). As well, another knob might change the number of times a card is drawn and replaced or not replaced.

In the case of the "not replaced = replaced problem," we decided that the knob our intuition pump needed to turn was the one that controlled the circumstance of the first card drawn. If we turn this knob counterclockwise from Problem B (giving us problem B^0), we change the circumstance of the first card drawn being placed face down beside the deck to being placed face up beside the deck. As we consider this problem, we soon realize that if the card is placed face up, we now know whether the card is a king or not. As in the original problem B, there are two cases: the first card is a king or the first card is not a king. What is different in problem B^0 is that this time the sum of the two resulting case probabilities (3/51 and 4/51) is 7/51, which is not the same solution as Problem B or Problem A, both of which are 4/52. In this case then, not replacing the first card has resulted in a different probability, which intuitively we expect when the first event involves not replacement. The next question to be considered is "what if this same knob is turned in the other direction," and it officially begins our engagement with an intuition pump.

ENGAGING IN THE INTUITION PUMP

Click. Problem B^2. I grabbed a deck of cards from the cupboard upstairs, removed the jokers, shuffled the deck, and then removed one card from the deck. Just as I was about to place the card face down on the table my phone rang. It was my oldest brother and his wife checking in on me, and we had a great phone call. An hour later, when I hung up, I went back to the table to consider the problem of the card being placed face down beside the deck, but the card was not there. I looked for the card but could not find it. So, I decided that I would just have to work out the probability without having the card sitting there beside the deck. I soon realized that I would need to consider the two cases as I had in problem B, and that the solution was yet again 4/52. I then noticed that the card had fallen onto the floor and landed face down. I reviewed my calculations and realized that even with the card face down on the floor the probability for this (intuitively) independent events problem still had the same probability as the dependent problem A.

Click. Problem B^3. Once again, I decided to try to role play problem B. I shuffled the cards, drew the first card off the deck, and then my dog, Euclid, started barking. He desperately wanted to go outside, so I went and opened the door for him and followed him out. I noticed that the bird feeder was pretty much empty, so I went to the garage to get birdseed. I took down the feeder and proceeded to fill it and then rehung the feeder. By that point, Euclid was at the door, wanting to get back in. We both headed inside, and when I sat down at the table and looked at the cards, I realized that yet again, the card I drew from the deck was nowhere to be found. I even recounted the deck to check if I had mistakenly replaced the card, but no, there were only 51 cards in the deck. Then, my thinking was, "hey, there are only 51 cards in the deck, so the probability will be different this time," but it was not. I still had to consider whether the (now missing) card that I drew was a king or not a king to find the probability that the next card I drew would be a king, 4/52. I lifted my head to ponder the situation and looked out my picture window and there was the card, sitting on the picnic table, face down, that I had used when filling the bird feeder. It was starting to rain as I calculated the probability again, this time knowing where the card was, but not what it was. Yet again, the solution was the same as problem A (the independent events problem) despite this new problem circumstance being clearly (intuitively) one involving dependent events.

Click. Problem B^4. The card I left outside dried (remarkably flat and indistinguishable from the other cards), and I was able to continue to use the same deck of cards. I shuffled the deck, drew a card, and then I saw an important form that I had forgotten to mail. I quickly got an envelope, wrote the mailing and return addresses on it, put a stamp on the envelope, put in the folded form, sealed the envelope, grabbed my car keys, and headed to the post office. I got the letter into the mail at 4:55, just five minutes before the mail was to be picked up. Relieved that I had gotten it off before the weekend, I went home and started to make supper. It was later that evening that I saw the deck of cards sitting on the table, and I realized that I had forgotten to finish solving the problem. I went over, and once again the card I had drawn was nowhere to be seen. A week later I received a phone message from an employee of the company to which I mailed my form. They had received the form but were wondering if there was any significance to the card. It really didn't matter anyway, when I did the calculation I got the same probability as before, 4/52. The solution was still the same as the problem with independent events, even though I knew (intuitively) the problem clearly involved dependents.

DISCUSSION

In the above intuition pump, the knob that we chose to turn controlled the circumstance of the first card drawn (replacement or not) and its location. The question that we were investigating was why what intuitively we believed to be dependent events (drawing of two cards with replacement of the first) was giving the same answer as the replaced independent events scenario (drawing two cards with replacement of the first). What we found (using some absurdity) was that adding complexity to the circumstance of the first card drawn when it was not replaced did not change the probability. That is to say, we found that in relation to probability, not all context is of consequence, including not replacement. Therefore, we found that our learned intuitive association of replacement with independence and non-replacement with dependence was not valid. In fact, it is only the case of non-replacement combined with identification of the result of the first event that leads to dependence between two events. In other words, only if the card is drawn, looked at, and not replaced is there dependency between the two events of the cards being drawn and regardless of where that card might be placed outside of the deck of cards.

The use of the intuition pump helped us, as mathematics educators and researchers, to develop a clearer understanding of the concept of "without replacement," and problematized frequently promoted intuitions in mathematics classrooms regarding the impact of "without replacement" on the size of the sample space and probabilities. We are next planning to take this intuition pump into our mathematics education courses for pre-service teachers and professional development experiences for in-service teachers to determine what effect it has upon the intuitions of these two groups.

THE INTRODUCTION: A REVISITATION

What the intuition pump exercise has revealed is that replacement and not replacement do not guarantee either independence or dependence (respectively); rather, what is seen (or not) combined with replacement or not replacement determines independence or dependence. For this reason, we now suggest that an expanded set of "a bag with marbles" problems be used to introduce the concept of the independence or dependence of multiple-event probability problems to students:

1. A bag contains 5 red marbles and 5 blue marbles. A marble is drawn from the bag, is not seen, and then replaced. A second marble is then drawn. What is the probability that the second marble is blue?
2. A bag contains 5 red marbles and 5 blue marbles. A marble is drawn from the bag, is seen, and then replaced. A second marble is then drawn. What is the probability that the second marble is blue?
3. A bag contains 5 red marbles and 5 blue marbles. A marble is drawn from the bag, is not seen, and then not replaced. A second marble is then drawn. What is the probability that the second marble is blue?
4. A bag contains 5 red marbles and 5 blue marbles. A marble is drawn from the bag, is seen, and then not replaced. A second marble is then drawn. What is the probability that the second marble is blue?

Of these problems, only the fourth one involves conditional probability in which the second event's probability is dependent upon the outcome of the first. This contradicts the commonly learned intuitive association of independence with replacement and dependence with not replacement, according to which both problems 3 and 4 would be believed to involve dependent events. Apparently, in this case, seeing is believing (your learned intuition).

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