THE PROBLEM WITH DECIDING IF ORDER MATTERS

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Does order matter? Such a simple question, yet students (teachers, and others) continue to struggle with deciding whether a probability or combinatorics problem involves permutations or combinations. This paper discusses findings from an ongoing study of pre-service secondary teachers (majoring or minoring in mathematics) who were given two related problems: determining the number of three-cube high towers possible when using only two colours of cubes and the Jane and Dianne Task (Chernoff & Russell, 2012). The question "Does order matter?" became problematized within this study when the participants used this question to interpret how they read the three-child family problem. What follows is a discussion of these two tasks and this specific question.

Similar to the work of Abrahamson (2009), the ongoing study discussed in this paper explores how students think and learn about binomial situations within contexts involving combinatorics and probabilities. Similar results emerged in this study in that the participants had some intuitive beliefs about such situations that were mathematically correct, as well as others that were not correct. Of specific focus in this paper is one such mathematically incorrect intuition may well have been school-constructed. As well as identifying a potential linguistic reason for this incorrect intuition, this paper proposes a solution that builds upon a correct intuition held by participants by redirecting some of their thinking through specific language choices.

THE TASKS AND PROGRESSION

In this study, a progression of two tasks were used to investigate the participants' binomial thinking in relation combinatorics and probabilities. The first task, and its extensions and dalliances, was investigated over two one hour and fifteen minute class periods (separated by one day). The second task, the Jane and Dianne task, began one week after the towers tasks had been completed. How these tasks evolved and how they connect to each other is now briefly described.

The two-coloured towers task

This task began within my first teaching (fall, 2014) of a third year secondary mathematics education course that is mandatory for all mathematics majors and minors within a secondary education program. It has subsequently also appeared (and re-appeared) in two other courses (one pre-internship and one post-internship) that I have offered to a subgroup of the same students.

The initial question I posed was: "How many three-cube high towers can you build using only two colours of linking cubes," and our primary focus in class was comparing and discussing the ways in which individuals solved the problem. With each iteration of this task, I started adding additional questions, such as "How many four-cube high towers can be built using only two colours of linking cubes?" and "If you were to build all 1-cube, 2-cube, 3-cube, 4-cube, and 5-cube high towers using only two colours of linking cubes, how many sets of linking cubes would need to be purchased?" The tasks were engaging for the pre-service teachers, and led to many student generated discussions and further questions, such as "what if you could use three colours?" Even within the first enactment of this task, the class "stumbled upon" Pascal's Triangle, the binomial expansion theorem, and the formula for determining binomial probabilities.

The Jane and Diane Task

The second task in the progression within this study, initially used by Chernoff and Zazkis (2011) to explore pre-service teachers' pedagogical approaches to the teaching of probability, and then modified slightly in 2012 to investigate pre-service teachers probabilistic thinking (Chernoff & Russell, 2012), was a natural follow-up to the task progression stemming from the two-colour towers task. As used in this current study, the task read as follows:

> What is the probability that a three-child family has two daughters and one son?
> *Jane's explanation:* Out of the four possible outcomes (3 daughters; 2 daughters, 1 son; 1 daughter, 2 sons; and, 3 sons) only one outcome (2 daughters, 1 son) is favourable, so the probability is one-fourth.
>
> *Dianne's explanation:* Out of the eight possible outcomes (daughter, daughter, daughter; daughter, daughter, son; daughter, son, daughter; son, daughter, daughter; daughter, son, son; son, daughter, son; son, son, daughter; son, son, son) only three outcomes (daughter, daughter, son; daughter, son, daughter; son, daughter, daughter) are favourable, so the probability is three-eighths.
> ________________________ 's explanation is correct because.....

The Jane and Dianne task, like the three-cube towers task, involved a situation in which there were three repeated independent events – the birth of a child and the selection of a colour, respectively. The purpose of bringing this task in after the study of the 2-coloured towers was to look for transference of knowledge from the one situation to another (isomorphic) situation.

RESULTS

Although the results provided herein are specifically based upon the data collected in the fall of 2017 through the formal study, it is worthy to note that the findings reflect the informal ones observed since the first offerings of these two tasks back in 2014. Below, a brief discussion of the participants involved in the study and a presentation of a relevant portion of the data collected will be presented.

Participants

The participants in this study were the entire class cohort of 11 mathematics majors and minors in the first term of their third year of a secondary mathematics education program. Seven of the participants had directly entered into the education program, putting them in their third year after graduation from high school. The other four participants were returning or continuing students. One had completed a mathematics degree at least ten years prior to enrollment in the education program, another had just completed a chemistry degree, and the remaining two were new Canadian citizens with varied work and post-secondary experiences, but none in education.

Results from the Three-Cube High Tower Task

As mentioned previously, there were many branches of exploration and learning that emerged from the initial three-cube high tower task, but the one relevant to this particular paper relates to how the students interpreted the combinatoric nature of the task. For the ease of reading, this portion of the paper will assume that every example given involved the students using yellow (Y) and green (G) for the two colours. A tower that has a green linking cube on the bottom, and yellow linking cubes in the middle and on the top is represented by GYY. A tower with a green cube on top and two yellow cubes below it is represented by YYG.

10 of the 11 participants immediately went to building eight possible towers: {YYY, YYG, YGY, GYY, YGG, GYG, GGY, GGG} (1). The remaining participant, Jeremiah (pseudonym) initially built four towers: {YYY, YYG, GGY, and GGG}, but upon observing the work of others in the group, he added four more towers to his solution. Upon noticing this change in the participant's answer, the instructor (researcher) proceeded to engage Jeremiah, and the rest of the class, in a discussion about this change. That exchange is given below:

> *Instructor:* Jeremiah, why did you change your answer?
> *Jeremiah* (pointing to the four other participants at his table): They told me that the order matters.
> *Instructor:* Does the order matter?
> *Jeremiah:* I didn't think so.
> *Susan* (pseudonym): It does, because you can see the difference.
> *Jeremiah:* I just don't know if the difference matters.

Susan (pointing to her YYG, YGY, and GYY towers): Of course it does, otherwise, how would you know which tower to go in to if you were told it had two yellow and one green floor?

Jeremiah: Oh, I see now.

Further discussion around this question, including the instructor challenging the participants twice more to justify why the answer could not be a set of only four towers failed to change any of the participants opinions that the order mattered. Part way through the discussion, Noah (pseudonym) noted that what was being said was that the towers situation was one that involved permutations rather than combinations. At that point, Monique, Meredith, and Sam, all of whom were from the French immersion education program, commented that this was a distinction in terminology that they were not familiar with within the French language. The other participants then defined the two words for the other three, emphasizing the distinction between whether the order mattered or did not matter within a given context.

The participants were then asked to answer questions relating to the probability of certain towers being built, such as a tower with three green floors or a tower with only one green floor. In all cases, the participants referenced the towers they had constructed directly or an alternative representation of that sample space, such as given in (1).

Results from the Jane and Dianne Task

Interestingly, where, in the tower task all but one of the students had immediately decided (correctly) that the problem situation involved permutations (even if they did not initially use that term), in the Jane and Dianne task, all but one of the students immediately decided (incorrectly) that Jane was correct because the "order does not matter."

The majority of the participants were very animated in their arguments for their decision. Four of them stated: "the question doesn't say that the order matters, so it must not." Four other participants argued that "the order just doesn't matter," but were unable or unwilling to provide a justification for their stance. The participants engaged in varying levels of role-play to demonstrate their reason for their choice, an example of which is provided below:

Mark: So, I'm out collecting data about whether families with three children have two girls and one boy. I go to a door, I ring the doorbell.

Katie (pretending to open a door)*:* Hello Mr. Survey Man.

Mark: Do you have three children?

Katie: Why, yes I do.

Mark: And what genders are they, if you don't mind me asking?

Katie: Not at all. I have two girls and a boy.

Mark: Thank you. Have a nice day.

Katie: You too.

Mark: See, I don't need to know what order the genders came in to know if they are to be counted as what I want as an outcome.

Again, the participants eventually started to bring in mathematical terminology, in particular the words combination and permutation, as they started to refine their arguments. Moreover, even when the instructor brought out a reserved set of constructed two-colour, three-cube high towers from the previous classes and said "what if Y stood for girl and G stood for boy" the students continued to argue that Jane's explanation was correct, because in the case of the family, order does not matter, but in the case of the towers, order matters.

ANALYSIS AND DISCUSSION

The data from this progression of tasks presented above relates to previous research of pre-service teachers thinking about combinatorics and probability. Although in the two-colour towers task the students are positive that the situation is one involving permutations, they are (as was also reported in Chernoff & Russell, 2012) convinced that the three-child family task is one involving combinations, or as the participants consistently communicated, in the case of the two-colour towers, "order matters," while in the case of families, "order doesn't matter." Moreover, the participants all worked very hard to remain convinced that in families order doesn't matter; whereas, questioning order in the case of the towers was simply dismissed by the participants.

Even when presented the physical model of the linking cubes, the participants did not recognize the cognitive dissonance between their two conclusions.

Of further interest is the emergence of "rules" that the students had either learned or created, such as "if the question doesn't refer to order, then it must not matter" and "if the question doesn't imply seeing the outcomes, order doesn't matter." The first of these two rules has been discussed in previous literature (Chernoff & Russell, 2012), while the second adds an interesting layer to the possible roles of visualization within probabilistic and combinatoric thinking.

As the discussion, and mostly one-sided debate, continued, a potential contributing factor to the issue started to emerge, and that was the word "matters." The data above seems to imply that when charged with deciding for themselves, and not using a standard rule like the two mentioned above, the participants were deciding upon whether order "mattered" in subjective ways. This can be seen when Mark said he did not need to know the order of the genders in the family to know if he should count the family as having two girls and one boy. In a similar way, but coming to the opposite conclusion, Susan's argument that she needed to know the order of the colours in the tower so she knew which one to go to, was also based upon her assumed (subjective) need to be able to distinguish the towers. Nothing in the original questions actually directed these two students to make these kinds of conclusions. It is this particular portion of the data which raises the question of whether the problem in distinguishing between permutations and combinations might be, at least partially, housed within the question students learn to ask, "does order matter," and more particularly with the word "matter."

CONCLUSION

To explore the above hypothesis, the participants in the study were asked to consider the question "does order exist" in relation to the two tasks. At first, the students struggled with changing the question, particularly with respect to the three-child family task, but when asked "could different orders exist within the genders of the families, whether the family has two girls and one boy or not" the participants all agreed that they could, and that even though the participants felt they did not need not know the gender order within a given family, they would count any family that could be described as having two girls and one boy, regardless of the order. At that moment, the long-sought clarity of understanding started to emerge; however, many of the participants quickly reverted to answering: "does order matter" in other isomorphic problems unless they were explicitly asked "does order exist?" When this second question was part of the problem, all of the participants correctly determined whether the situation involved combinations or permutations, when it wasn't, the majority went back to the incorrect answer.

It appears as though, somewhere in their previous mathematics experiences, the participants in this study had very effectively (and almost permanently) learned that the question "does order matter" was key to distinguishing between permutations and combinations in combinatoric and probability questions. What remains to be determined is whether, if students learn to focus on the question "does order exist" (or a similar variation on the theme), will future participants (students, and teachers) be able to successfully distinguish between situations involving permutations from those involving combinations, and be able to provide correct arguments for their choices.

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