STUDENTS' CONVEYED MEANINGS FOR PROBABILITY

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Probability is crux that allows statistics to be a useful tool for many fields. Thus, the meanings that students develop for probability have the potential for lasting impacts. This report extends Thompson's (2015) theory of meanings through the notion of conveyed meaning: the constrained implications that a receiver attributes to the sender's statements. A student's conveyed meanings give insight into his/her initial and/or dominant meanings for a particular idea. This report shares the results of two studies: one examining ~114 undergraduate students' conveyed meanings for probability after they received instruction as well as their instructors' conveyed meanings. The other on ~134 students conveyed meanings before and after instruction intended to combat the worrisome presence of circular conveyed meanings of probability.

INTRODUCTION

Probability is the crux that allows statistics to be a useful tool for many fields. In particular, probability is the result of centuries of work towards one goal: the quantification of uncertainty. Since before the 1600s mathematicians, philosophers, logicians, and statisticians have attempted to resolve questions where uncertain outcomes dominate (Weisberg, 2014). Over the course of history, many scholars have engaged in what Thompson (2011) calls quantitative reasoning and quantification. In settling what a measure of uncertainty means (along with how to get a measure and what is meant by measuring uncertainty), scholars have taken different paths and arrived at their own meanings for the same notion, probability. Laplace considered the ratio of the number of outcomes of interest to the number of all possible outcomes under the assumption of "equally likely" outcomes (Weisberg, 2014). Von Mises (1981) considered repeating some process indefinitely to build a collective and that the limit of the relative frequency of an event of interest was the probability of that event. Kolmogorov's (2013) axiomatic, measure-theoretic approach has become the gold standard for probability theory. De Finetti (1974) and Savage (1972) regarded probability as dealing with measuring the amount of belief that an individual had for a particular outcome's occurrence that they called "subjective" or "personalistic" probability.

The opening statement of this paper is one that hardly any practitioner of statistics will disagree with. Regardless of which school of probability you ask (i.e., Frequentist, Bayesian, Conditional Frequentist, etc.) each acknowledges that the central ideas of probability allow us to move beyond merely describing a data set to using the data set as evidence for supporting or refuting claims. The members of these schools of thought have already carried out the quantification of uncertainty, something that students have yet to undertake. How practitioners think is often vastly different from how students think before, during, and after instruction. Kahneman and Tversky (1974, 1982) described how individuals will use different heuristics when making judgments under uncertainty. For example, how representative an event (sample) is to the parent process (population) to can influence a person's estimate of the probability of the event. Another heuristic that they found people use to measure uncertainty centers on the ease (or lack of) with which a person can imagine the event occurring; the more "available" an event is for the person to imagine, the larger the probability (the less uncertainty) there is for that event. Konold (1989) found that for some individuals, their way of thinking about probability did not match the use of heuristics nor was their thinking consistent with the schools of probability. Rather, these individuals appeared to view the goal of uncertainty to be the prediction of the next result; Konold referred to this way of thinking as the outcome approach to probability. Students also have a tendency to view events as equally probable when they do not perceive the many ways a compound event might occur (Lecoutre, Durand, & Cordier, 1990). Lecoutre et al. found that students and adults view the event of getting a five and a six as having the same probability as getting two sixes when rolling two dice. They hypothesized that not recognizing that event of (5, 6) is comprised of two smaller events. This way of thinking across multiple events is

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what they referred to as the equiprobability bias. Fielding-Wells (2014) found that when trying to pick the best card for playing addition-bingo, Year 3 students (7-8 years) operated as though all of the sums of the numbers 1 to 10 were equally probable. Saldanha and Liu (2014) reviewed much of the literature on students' understandings of probability and proposed that a key conceptual scheme for understanding the measurement of uncertainty is a stochastic conception. They define a stochastic conception as "a conception of probability that is built on the concepts of random process and distribution" (p. 393). They argue that in the quest to support students developing coherent probabilistic reasoning, instructors need to conceive of probability as ways of thinking rather than skills and design curriculum that supports this. While this review of the literature is brief, already apparent is the fact that the quantification of uncertainty is challenging. Individuals of all ages and backgrounds struggle just as mathematicians, statisticians have to construct a meaning for probability.

THEORETICAL BACKGROUND

To investigate my questions, I turned to the theory of meanings that Thompson and Harel have devised (see, Thompson, 2016; Thompson, Carlson, Byerley, & Hatfield, 2014). *Meaning* refers to the space of implications which includes actions, images, and other meanings that results from an individual assimilating some experience and thereby forming some understanding of that experience (Thompson, 2016). The space of implications, that is the meaning, is the inference that accompanies assimilation (Jonckheere, Mandelbrot, & Piaget, 1958). Central to the radical constructivist perspective is the belief that every individual builds his/her own knowledge through repeated experiences (von Glasersfeld, 1995). The meaning that an individual imbues an experience with is a product of their constructions. Thus, an individual's meanings are intensely personal, and researchers do not have access to that individual's meanings. This problem is a familiar one to mathematics education researchers. Thompson (2013) tackled a similar question; if meanings are entirely within individuals, then how can people learn a meaning from someone else? Thompson's found an answer by turning to the notion of intersubjectivity and Pask's conversation theory. Intersubjectivity hinges on an individual having a mental image of another person that is free to think like and not like individual (von Glasersfeld, 1995). From Pask's theory, Thompson (2013) highlights that a conversation is more than just a verbal exchanges; conversation also includes all of the participants' "attempts to convey and discern meaning" (p. 63). As two individuals have a conversation, (let's call them Alexa and Siri) they must each keep in mind not only what they wish to communicate but also how the other person might interpret her words/actions. Both of them build a model of the other person. Thompson's work provides an answer for how the conveyance of meaning from one individual to another might occur. When Alexa wants to communicate something specific to Siri, she provides communication which Siri then assimilates to her schemes. Through that assimilation, Siri imbues that experience with a meaning that stems from two sources. The first source is her own meanings; the second is what she knows about Alexa. The meaning that Siri gives to Alexa's communication is the *conveyed meaning.* A conveyed meaning is the set of implications that a receiver attributes to the sender's message constrained by 1) the receiver's de-centering and 2) the receiver's belief that the sender made an honest effort to convey his/her thinking. These constraints lay the groundwork for intersubjectivity and keep both participants in the picture. While Alexa's conveyed meaning might not be a perfect reflection of her actual meaning, this is how Siri understood Alexa and is the basis for which Siri's response. The notion of conveyed meaning is useful in education in several ways. First, we can use this notion in research to attempt to discern what meanings our students have constructed for various topics. Second, we can use this notion in the planning of lessons. By trying to answer the question of "what have I conveyed to my students?" we can engage in de-centering and design meaningful conversations. This second use is easily extended to a third focused on the generation of curriculum materials such as activities and textbooks. With textbooks, we can imagine to types of conveyed meaning; the first being what the authors conveyed to us and the second being what the authors conveyed to our students.

By examining students' responses, we can characterize those responses by the meaning conveyed. However, to compare categories of conveyed meanings there must be an aspect of the theory that deals with productiveness of the meanings. Thompson (2016) proposes that productive meanings are those meanings that provide coherence to ideas that students have and those meanings which afford students a frame that supports the students in future learning. *Productive meanings* are clear, widely applicable (within reason), and entail an awareness of and need to explicate any assumptions. Notice that the usefulness of a meaning is part of a productive meaning. I take a *useful meaning* to be any meaning that allows a student to meet some performance or learning goal. Consider the following meanings for the associative property: A) move parentheses around vs. B) the choice of which of two structures to impose ($[a+b]+c$ or $a+[b+c]$) does not change the result. Meaning A is useful; students can get correct answers; however, this meaning does not necessarily help students when there are more than three terms. However, meaning B is a productive meaning and useful.

STUDY ONE

The first study (N. J. Hatfield, 2016) sought to answer the following two research questions:

- What meanings do students convey for probability after they received instruction?
- Are there differences in the students' conveyed meanings based on which instructor they had?

I will only report on the first question here.

Methods and Setting

To do so, I conducted an observational study at a large, public university located in the Southwestern region of the United States during the Spring 2014 semester. Students enrolled in an introductory statistics course aimed at life science majors took a written survey during a regular class period after they had received instruction about probability. The course had four sections, taught by three instructors. Instructor A was a Ph.D. Math Education student who had not previously taught the course, Instructor B was a Senior Lecture (MS-Statistics) who had taught the course many times and served as the course coordinator, and Instructor C was as Ph.D. Statistics student who had taught the course the previous semester. Instructors B and C (B had two sections) followed the textbook (*Statistics for the Life Sciences* (Samuels, Witmer, & Schaffner, 2012)) closely. Instructor C made use of many of B's materials. Instructor A departed from the textbook, and instead focused on designing a course that sought to foster students' construction of productive meanings. Each of these instructors also responded to the written survey. The two questions students and instructors responded to were:

- Question 1: How do you think about probability? That is, how would you explain probability to another person?
- Question 2: Consider the following statement: The probability of observing a value of 4 when looking at the product of two dice is 3/36. How should someone think about (interpret) 3/36 given the above statement?

To analyze the responses I used a grounded method consistent with that described by Strauss and Corbin (1990). I initially used open coding for the responses and then I made use of an axial coding system. I used the axial codes in my analysis that follows. This approach partnered with the theoretical perspective is also consistent with the general methodology used in the development the Mathematical Meanings for Teaching Secondary Mathematics instrument described in Thompson (2016).

Results

Table 1 shows the frequency of responses that fall into these categories for Question 1. Overwhelmingly, 89 students (78.1%) gave a response conveyed a circular meaning. Nineteen students (16%) appear to think about probability in terms of frequency/relative frequency. Of these students, 15 thought about probability as the long-run relative frequency of some process.

Table 1. Students' Conveyed Meanings for Question 1

L.R.R.F.	Classical	Frequency	Prediction	Circular	Other	total
15	0	4	3	89	3	114

As shown in Table 2, a majority of students interpreted the probability value (Question 2) as being about a fixed number of rolls of the dice and a fixed number of 4's (40.5%). Only 17.1% (19) of the students thought about 3/36 as representing the percent of the time we would see a product of 4.

Fifteen students appeared to use a "classical" way of thinking, while 19 just substituted "chance" for "probability".

Table 2. Students' Conveyed Meaning for Question 2

L.R.R.F.	Classical	Relative Frequency	Fixed Number	Circular	Other	total
19	15	0	45	19	13	111

A natural question that follows from the previous two questions, is how do the students' responses to each question relate to one another?

Table 3 shows the two-way contingency table for students' responses to both questions. The vast majority of individuals who appeared conveyed that probability is the long-run relative frequency of some event also conveyed that a given probability value as the percent of the time we would see some event happen in the long run. The majority of students who conveyed that 3/36 as being two numbers separated by a bar (either Classical or Fixed Number) or as a "measure of chance", also conveyed a circular meaning for probability. The wide range of interpretations given by students with a circular meaning is not surprising. Given that the students' meaning for probability appears related to a word-exchange, the students would need to draw upon some other meanings to help make sense of the value 3/36. All but one student, who explained 3/36 as the "chance" of getting a product of 4, gave responses that indicated a circular meaning to Question 1.

Table 3. Students' responses to Question 1 by their responses to Question 2.

	Percent of the Time	Classical	Fixed Number of Rolls	Chance	Other	total
L.R.R.F.	12	1	0	1	0	14
Frequency	1	0	1	0	2	4
Prediction	2	1	0	0	0	3
Circular	4	12	42	18	10	87
Other	0	0	1	0	2	3
total	19	14	44	19	18	111

I will quickly mention that three students did not answer Question 2, thus the reduction in the grand total to 111. Taking these results together, we can see most students conveyed meanings that are less productive than what they could have. The overwhelming percentage of students who initial conveyed a circular meaning for probability for Question 1 is worrisome. The prevalence of such conveyed meanings and whether or not this conveyed meaning is truly indicative of the students' actual meanings require further research.

STUDY TWO

The purpose of the second study was to examine the impact that having targeted discussions about the distinctions between probability, chance, and likelihood would have students' conveyed meanings.

Methods and Setting

The second study focuses on two courses taught at the same large, public university located in the Southwestern region of the United States of America during the Fall 2017 semester. The students here enrolled in an introductory statistics course used by many different majors. While this course has a separate course code from the course used in the first study, both courses cover the same content; life science majors may use either course to fulfill program requirements. The second study looked at two sections; an in-person, 15-week long course (62 students), and a fully online, 7.5-week long course (110 students). Both sections involved were taught by the same instructor who identifies as a Mathematics/Statistics Education researcher. Given the prevalence of circular conveyed meanings from the first study, the instructor developed materials and class discussions to introduce the distinctions between probability, chance, and likelihood. At the start of the course, students responded to a series of questions meant to gage the meanings they were bringing with them on core ideas such as probability, randomness, distribution, and variables. The set of questions on probability included the two questions from the first study along with eighteen additional questions that looked at their conveyed meanings for probability, chance, likelihood, and odds. At the end of the course, students responded to several questions including the second question from the first study. Students responses were coded using the same set of axial codes as in the first study (N. J. Hatfield, 2016).

Results

Table 4 shows the classification of students' conveyed meanings for probability at the beginning of the courses. The totals lower than the course enrollments as three individuals did not complete the assessment in the in-person section and thirteen individuals likewise for the online section. Approximately 40% of the students came into the course conveying circular meanings for probability. Another 23.7% of students conveyed that probability told us the exact number of times an event will happen given a set number of trials. No student started the course by conveying the probability was the long-run relative frequency of observing some event when imagining repeating the process indefinitely. Two students who did bring up relative frequency but did not convey repetition.

Table 4. Students' Conveyed Meanings for Probability at Start of Course

	L.R.R.F.	Classical	Relative Frequency	Fixed Number	Circular	Other	total
In-person	0	11	1	18	24	5	59
Online	0	28	1	19	37	12	97
total	0	39	2	37	61	15	156

At the end of the course students once gave responses to question meant to elicit their meaning for probability (

Table 5). There was one student in the in-person course who did not take the final exam; 41 students did not take the final exam in the online course. Nearly 65% of all students gave responses that conveyed that their meaning for probability was focused on long-run relative frequency as opposed to circular conveyed meanings.

Table 5. Students' Conveyed Meanings for Probability at End of Course

	L.R.R.F.	Classical	Relative Frequency	Fixed Number	Circular	Other	total
In-person	43	6	4	1	4	0	58
Online	58	5	0	4	9	0	76
total	101	11	4	5	13	0	134

DISCUSSION

The first study reported here demonstrated that a large majority of students (78.1%) conveyed circular meanings after receiving instruction on probability. A hallmark of the impact of education's impact should be the movement of students' from less to more productive meanings. For such a large proportion of students to have unproductive meanings suggest that something problematic was going on. The second study sought to see what would happen when instructional design took seriously the idea of conveyed meanings and targeted circular conveyed meanings. The second study address limitations in the first study by using a wider battery of questions as well as capturing students' conveyed meanings from the start of the course. Most importantly, the second study demonstrates that the circular conveyed meanings can be addressed through targeted interventions that include class discussions, homework problems, and instructor-generated materials. Such instructor interventions appear to work for students attending in-person classes as well as fully online classes.

While I make no generalizations from these studies to the wider population, together these studies raise worthwhile questions for us to explore in Statistics Education. From the second study, about 40% of the students walked into the course with circular meanings for probability. If we assume that there is little difference between the students in two studies, then for the proportion of students conveying circular meanings to increase to 78.1% would suggest that students are *learning* circular meanings for probability, chance, and likelihood. Hatfield (2016) notes that this could be a result of instructors' own meaning acting as limiting factors as well as influence from curricular materials such as textbooks. The power of the idea of conveyed meanings is not restricted to just probability. We can apply this construct to many other ideas such as stochastic processes, stochastic variables, and the all-important concept that is distribution. If we truly want to help our students construct productive meanings for statistical concepts, then we must begin to pay careful attention to meanings that we convey and that our materials support students in developing.

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