CLASSROOM ACTIVITY IN SECONDARY SCHOOL FOR INTRODUCING CONDITIONAL PROBABILITY

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A formative activity based on a paradoxical situation, "Three cards game", was proposed for introducing the notion of conditional probability to secondary school students (14-16 year olds). We applied this activity in a classroom of 26 Colombian learners and analyzed qualitatively their responses, comparing them with previous researches. This intentional sample showed good performance except for a few students who still made mistakes at the end. We concluded that this activity was successful because the majority of students built notions as conditionality, dependent and independent of events, and all of them were highly motivated during the class.

THE PROBLEM

Conditional probability underlies our daily formal and informal decision making; in most cases assertive decision making in uncertain situations is based on conditional reasoning (Arteaga, Batanero, Contreras & Diaz, 2014). Many people are not conscious about the relevance of auxiliary information in making decisions or the influence of personal beliefs in allocating probabilities. For this reason, since the 1990's the curriculum of different countries (as MEN, 1998) included competences linked to comprehension of this concept (or at least of its notion); this curriculum highlighted that a citizen must be well informed for making the best decision. In the Colombian curriculum, MEN (2006) it is suggested that conditional probability reasoning should be developed in the last scholar years of obligatory Secondary School (beginning when students are 14 years old).

Probability and conditional probability are concepts which are hard to understand for many people and some authors have mentioned that the problem arises in the diversity of meanings associated with them and in the comprehension of uncertain events. For example, Batanero, Henry and Parzysz (2005) analyzed the concepts of chance and probability from epistemological and historical points of view. This ended with implications for teaching and learning linked to philosophical controversies surrounding the nature of probability. Borovcnik (2012) revised this from a conceptual point of view with multiple perspectives of conditional probability, including educational difficulties and solving strategies. He ended up organizing an agenda for further educational research beyond this issue.

In this paper we propose and analyze a classroom activity with the aim of introducing the notion of conditional probability to secondary school students (14-16 year olds). Firstly, we summarize academic literature related to teaching and learning this topic. Secondly, we present the proposed activity, briefly describing our application to a classroom of Colombian students and a certain relevant results. Finally, we summarize the main conclusions of our work.

BACKGROUND

Research about comprehension of conditional probability

According to Jones and Thornton (2005), understanding conditional probability had been interesting for psychological and educational researchers (we will mention only some of them). Falk (1986) observed systematic biases in probability reasoning in adults, highlighting misconceptions and fallacies as conjunction fallacy or confusion of the inverse. Totohasina (1992, cited by Contreras, 2011) conducted a study with French students (16-17 year olds) in two phases, firstly he observed intuitive strategies and then applied instructional activities using two representations; here, the tree diagram was more effective than the two-way table in solving problems of conditional probability.

The authors concluded that it is common to confuse conditional probability with probabilities of other events such as marginal probabilities, conjoint probabilities and other conditional probabilities. Furthermore, it's complexity underlies the notion that recognition of statistical dependence between events is part of abstract thinking.

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Paradox for learning probability and conditional probability

Lesser (1998) promoted the use of counterintuitive examples (as paradoxes) for statistics learning as a support of constructivist pedagogy because of "engaging those beliefs to yield deeper understanding and supporting the role of the teacher as facilitator of those *what if* questions" (p. 10). Additional benefits to students include improving their abstraction ability and awareness of personal reasoning strategies. The author analyzed responses to his own Statistics Opinion Survey and students enrolled on three sections of service introductory statistics course before beginning classes; he concluded that interest and surprise about the statements are highly correlated (0,666). His primary recommendation was "to limit examples to those that actually occur in real life […] and that can be readily explained or explored by means other than analytic mathematics alone" (p. 12).

Batanero, Godino and Roa (2004) proposed activities based on counterintuitive examples, as suggested Lesser (1998), to assess comprehension of probabilistic concepts by prospective teachers. In particular, we focused on a modification of Bertrand's box, named "Three cards game", initially used in a psychological research conducted by Shannon and Weaver in 1949 to ask for reasoning when people made decisions.

Contreras (2011) summarized research about the relevance of using a simple paradox as a motivational situation in the classroom and its potential influence on student's comprehension of mathematical concepts. He described various paradoxes cited in academic sources and on web pages which showed possible difficulties when they were evolved in a teaching practice and analyzed didactical suitability, which included a detailed review of "Three cards game".

PROPOSED ACTIVITY

The goal of our activity is to build the notion of conditional probability in secondary school students (14-16 year olds). We selected as a paradoxical situation "Three cards game" because it is appropriate for teenagers and easy to play with cards made by themselves.

We took the ideas of Batanero, Godino and Roa (2004) and adapted them for our own purposes and objective population because these authors proposed these games to assess conditional probability reasoning in prospective teachers or university students. As we started to introduce this notion to younger students, we decided to give more explanations and opportunities to understand the game as playing and guiding students step by step helped them realize that the probabilities to win this game change when you have additional information.

Our activity has two sections (Figures 1 and 2). It begins with a verbal explanation of "Three cards game" (first paragraph of Figure 1) and with suggestions to illustrate it and bring it to life (second paragraph). The idea is that the teacher plays with all the students three times using cards made by the teacher.

> Let's look at the following random experiment:
>
> We have a box on the table and inside the box there are three cards. One card is blue on both sides, another card is red on both sides and the other card has one side blue and one side red. Someone randomly takes a card, this person (in our case, the teacher) shows one side to the players (in our case students). Each player tries to guess the color of the hidden side (the side visible to the first person but not to the player) and writes their prediction. After ten seconds the hidden side will be showed and each one will know if they guessed correctly. The person who correctly guesses the color of the hidden side wins the game and the experiment ends.
>
> To verify that everybody understands the experiment the game is played three times with a modification: instead of writing their prediction each person raises their hand when hearing their option, because the teacher will first ask aloud who predicts that the color of the hidden side is red and secondly who predicts that it is blue. Note that when the experiment has ended, the card is placed in the box again before the next trail and the experiment is repeated under the same conditions.

Figure 1. Translation of initial statement of "Three cards game" adopted for us.

The second section has three parts and six items (Figure 2) for pedagogical reasons.

The first part is composed of two items relating to intuitively or probabilistically the best strategy to win the game and should be responded to independently. The first item directly asks for a solution and their reason with two objectives: to know if the student has applied any probability notion or has answered intuitively, and to use as a base line to compare final responses. The second item asks for the use of the tree diagram to represent sample space (and each one of the stages).

The second part is based on experimentation and should also be responded to independently. The guide includes a table of results to register each time a prediction is made and if it was a success. After playing 12 times the third item asks to identify dependence among events, taking into account the results of previous runs (registered in a table). Here we expected that each student would recognize what event had the highest probability.

The third part has three items and should be responded to by groups of four students based on previous results and to discuss responses with partners. The fourth item is only informative and asks for a comparison of the results among partners. In order to introduce the fifth item which asks to identify the winning strategy, we expect that they select the most likely event. The sixth item has two questions about compute conditional probability, each one asking for a different conditioner event.

At the end of the class, we suggest a global discussion to answer the last question "What is the probability of giving a correct prediction?". We assume that they recognize the winning strategy as the more probable event because they are conscious of probability and that students linked their responses with the tree diagram built in the second item and created the restricted sample space.

First part

1. Do you prefer to choose the same color of the visible side or the other? Do you not have color preference? Why not?

Second part

2. Using a tree diagram, what are all the possible results of this experiment?

After verifying that the experiment is understood we play: we repeat this experiment 12 times (trails) and each tine that your prediction is the same color as the hidden side, you win one point. On the next table write your score and predictions (using B when you choose Blue and R when you choose Red)

TRAIL	1	2	…	11	12	Total
Color of visible side						
Prediction of hidden side						
Observed color of hidden side when it is showed						
SCORE (1 if your prediction was right, 0 the in other case)						

3. Did you use some strategy to win in the game? ______ If so, what was your strategy? ____
____

Third part

4. Make a group of four students (partners). Who gets the highest score? ______________
5. Taking into account your scores, what would be the best strategy to win in this game?
6. Taking into account the results of the table and/or tree diagram, answer the following questions:

a. If on the visible side the color is blue, what is the probability of the hidden side being red?

b. If on the visible side the color is blue, what is the probability of the hidden side being blue?

Figure 2. Translation of questions and instructions included in proposed guide

IMPLEMENTATION

Participants included 26 students between 14 and 16 years old (13 girls and 13 boys), who lived in a populous, poor neighborhood on the limits of the main city (almost outside) and with many socio-economic problems.

We applied our purpose in 2016 to an elective class (which is done in addition to regular classes) with the aim of improving random thinking in participants and abilities for handling data. Students' attendance is twice a week with each session around 90 minutes, for four months; at the beginning of the course, students only had intuitive notions of probability and randomness; the teacher followed problem solving methodology and used tools available on the internet.

Our experience was applied in one session of this elective class. To begin the teacher read carefully the game statement (Figure 1) and clarified doubts about it, after that each one read and wrote by themselves (first part of the guide in Figure 2). All the time the teacher was available to answer questions or to broaden explanations, for example some participants showed confusion in the second item (build experiment's tree diagram) about the amount of stages. Then the teacher read aloud again the first paragraph of Figure 1, highlighting step by step what are the experiment and gave them more time to respond.

For the second part, the teacher played the game 12 times so that each time they predicted what color would be on the hidden side of the card that the teacher had in her hand. Each time the teacher gave them around 10 seconds to write their personal prediction and showed the hidden side of the card for each student. The teacher would also let them know if they were a winner in this run because their prediction had hit the color of the hidden side. The instruction was to register each personal prediction in the table, but many students were enthusiastic and said aloud before writing their prediction; we do not know how this could affect the responses of their partners. The teacher played an additional five times which were not registered, because she noted that the students needed to reinforce their personal strategies to win.

To answer the third part of the guide, students worked in groups of four to discuss personal strategies because they had to identify the best strategy to win (item 4); as they were excited with the game, the teacher invited them to keep calm before continuing with the activity.

The final question, "What is the probability of giving a correct prediction?", was hard for them because the statement was confusing for some of them. Contrary to what we expected, some learners based personal strategy on beliefs, luck or intuition. As our aim was that they realized that using conditional probability was a source of ascertaining the winning strategy, the teacher helped them to focus their analysis on the tree diagram in the second item and to link it with observations on the table of the experiment' repetitions.

In the next class the teacher explained in a probabilistic way the winning strategy and developed other problems by referring to games (such as blackjack) supported with a tree diagram, which are not counterintuitive but are clearly compose experiments with dependence between stages.

RESULTS

The written responses of students were classified by categories and analyzed for each item. Now we show observed categories for three items (2, 3 and 6a) and summarize the main results; for more details, the reader can see Latorre (2016).

Observed categories for item 2

Most of the participants (88,5%, 23 out of 26) <u>correctly built the tree diagram</u> which was evidence that they recognized each stage of the experiment; after that they wrote sample space of this random experiment based on their diagram. Maybe this success was associated with careful repetition of the statement as was mentioned in the previous section, because initially they were confused with the instruction and asked the teacher to clarify the situation. Also their answers to item 1 showed that they thought the problem was a simple random experiment.

Only three students (11,5%) <u>built an incomplete diagram</u>, which showed partial comprehension of compose random experiment. Contreras (2011) justified these kind of errors as a misinterpretation of statement or failures on recognizing stages and its representation on the diagram.

Observed categories for item 3

Only two students (7,7%) <u>identified a strategy to win the game</u> without probabilistic arguments, they recognized dependence between the stages of the experiment. One of them wrote "As there are 3 cards, 2 have the same color on both sides, instead of there being only one card with a different color on each side". The predictions of this participant were right in 8 out of 12 trails.

Half of the participants (53,8%, 14 out of 26) <u>guessed the color of the hidden side</u>, they gave the right predictions in more than a half of the trails but said that they did not use any strategy. Maybe they intuitively answered or they had some ideas about patterns, but they were not confidence about their argument or did not know how to write it. While the experiment was repeated, the teacher listened to expressions such as "if red is shown you have to choose red because it is more probable" or "if blue is shown I select it because it has more chances". These phrases are evidence of the understanding of dependence between events, at least as a notion, but no one wrote them on the guide.

The rest of the participants (38,5%, 10 out of 26) <u>faulted most of the time</u>, they gave the right predictions in a few trails but in general said that they did not use any strategy. Only one of them said that they used a strategy "chosen by chance"; analyzing their responses on the table, it seems that this participant associated equiprobability as property of random experiment because their predictions showed a pattern to balance out the two possible outcomes. This is known as player fallacy, a kind of heuristic representativeness described by Kahneman, Slovic and Tversky.

Observed categories for item 6a

Half of the participants (53,8%, 14 out of 26) computed conditional probability based on the tree diagram. They responded correctly to item 2 and here recognized all the components of conditional probability. For example, "P(Red)=1/3 because there are 3 options of the visible side being blue and of these 3 there is one possibility of the hidden side being red". This student used wrong symbolic representation because they were not introduced to conditional probability before, so did not know the use of P(Red|Blue). The restriction of sample space and dependence between events were clear in the argument. Without probabilistic language, the reader can recognize each part of conditional probability; firstly, it exposed the conditioning event, the student focused on blue cards on the visible side; then identified the conditioner event with favorable cases only among this subset, the student looked for cards that had red in one side with blue on the other side.

Nine participants (34,6%) made a mistake related with probability of events on composed sample space, they responded correctly to item 2 and here wrote the correct fraction but with the wrong argument. They misidentified the involved events and their probabilities, the underlying mistake being on failures to recognize the composed sample space. One student wrote "1/3 because there are three red sides in the box"; this participant did not take into account information given in the statement and ignored dependence of events or missed out restriction. Other answers were "1/3 because there is only one card with colors blue and red"; it seems an example of equiprobability bias defined by Lecoutre and mixed with a simplification of the experiment, the student assumed that the experiment was only to take a card.

Only three students (11,5%) responded without explanation or process, they responded incorrectly to item 2 and here only wrote a fraction which is the correct answer (1/3).

When we compared responses between initial question and final ("What is the probability of giving a correct prediction?"), we observed the following: (a) Students selected the same color of a shown face in the first item, ratified it in the last item and the difference was that probabilistic arguments were mentioned. (b) Students expressed that they were indifferent to the choice of color in the first item, changed their perception in the last item and also preferred the same color of a shown face based on sample space restricted, dependence of events was identified on branches of the tree diagram. (c) Few students had trouble giving a probabilistic argument for select a color.

We consider as the main results: (a) that most of the students identified correctly the winning strategy which is to predict the color of the hidden side by choosing the same color of the visible side; (b) half realized that there was a probabilistic argument to select this strategy; (c) some students replicated procedures without conceptual comprehension; (d) others maintained their initial position (the hidden side is unpredictable because the game is random) and did not recognize that a winning strategy existed. Those who identified a winning strategy did it quickly, in the first item only two said they had a personal strategy; at the end 16 students said that, after they had discussed with their partners and played the game several times.

The success is partially because few students revealed some misunderstandings and made mistakes. Compose experiment caused confusion especially if elementary events are non-equiprobable. Equiprobability bias appeared in a few students and for them it is hard to comprehend that different results of a random experiment could have unequal probabilities. Maybe this mistake underlies a wrong idea: equiprobability is property of randomness.

CONCLUSION

The aim of this work was to design and apply an activity to develop a notion of conditional probability in Colombian students in ninth grade (14-16 year olds) by focusing on analyzing a paradoxical situation, "Three cards game". Our results showed that most of the participants built notions of conditional probability, dependence between stages on compose experiment and independence between successive trails of the same random experiment. Some of them understood conceptual and procedural aspects of them. Almost all realized the importance of this topic and broadened their probabilistic view, because at the beginning of the course they always used Laplace rule to compute probabilities in random experiments.

Bertrand´s paradox presented as "Three cards game" made it easier to develop this activity because our students were very enthusiastic and liked to win; they tried to successfully predict the color of the hidden side and to determine a winning strategy. For students it was interesting to discover that low-logical strategy (counter-intuitive) at the beginning of play would be the winner; it confirmed what Lesser (1998) proposed. Furthermore, most of the participants ended up having a probabilistic argument to support a winning strategy.

We noted that some participants needed more time to analyze the problem by themselves; when they played the game only a few students really used the results of their own predictions to create a personal strategy to win (guess the color). Moreover, we recommended playing more than 12 trails to avoid promoting a false idea of the small numbers law and to give them more elements to recognize the trend of the experiment by themselves. At the end few students showed misconceptions, as Falk (1986) recognized in adults, in particular equiprobability bias or confusion with compose experiment, especially if elementary events are non-equiprobable.

We verified the advantages of using different representations as observed by Totohasina (1992, cited by Contreras, 2011). The tree diagram is an effective tool to understand compose experiment, statistical dependence between stages and conditionality because participants visually identify conditioning and conditioner events through branches. However, some students did not recognize composition in the experiment, but also followed links to simple experiment.

Finally, we highlight that the teacher is very important for successful learning because students alone could be lost in the game and not grasp the objective of the activity. As such they would therefore not build a notion of conditional probability which is a complex concept that is usually very confusing, not only to teenagers also to adults.

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