REASONING DEVELOPMENT OF A HIGH SCHOOL STUDENT ABOUT PROBABILITY CONCEPT

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In this article the development of reasoning of a high-school student on the concept of probability is described from the inferences formulated as he solved three problems. An adaptation of Jones et al.'s (1999) framework was used to indicate the important characteristics in his reasoning. As a result, the difficulties faced by the student at different moments were: overcoming the law of small numbers, managing the variation in a convenient way, giving meaning to the quantification of the propensity of occurrence of an event, articulating the uncertainty of the individual outcomes with the long-run regularity of the relative frequencies, and using probability as a premise to formulate inferences. Three important elements to overcome these difficulties were: an informal knowledge of the law of large numbers, the reasoning with relative numbers and the partial institutionalization of the classical interpretation of probability when it was already informally used by the student.

INTRODUCTION

The level of understanding a subject reaches on a concept is associated with the meaningful and plausible inferences he is capable of formulating with it (Brandom, 2000). Concerning the concept of probability, the most important inferences are "the *unpredictability* of random phenomenon in the short-run [and] the *predictability* in the long-run trends in data (i.e., the law of large numbers)" (Sthol & Tarr, 2002, p. 321). In order to formulate such inferences it is necessary to make both the frequency and classic approaches of probability intervene and articulate. Besides, it is convenient to consider as well the subjective approach since it "is [more] consistent with the common everyday notion of the "likelihood" of an event. [Then] one of the challenges of making probability meaningful for students is to help them integrate their informal understanding of the likelihood of an event with the formal definitions they encounter in the classroom." (Dollard, 2011, p. 30). To elucidate this process of articulating, the framework elaborated by Jones, Thornton, Langrall and Tarr (1999) is suggestive, since the four reasoning levels proposed therein point out the characteristics of the cognitive progress of the students during the process of the quantification of the propensity of the occurrence of an event. In this study, the categories proposed by Jones et al. are used to organize the productions of one high-school student. It is described how he manages to articulate the different interpretations of probability during the process of formulation of probabilistic inferences.

CONCEPTUAL FRAMEWORK

Jones et al. (1999) propose a framework that describe and predict students' probabilistic reasoning (8-14 years old) based on four levels: *Subjective*, associated with a idiosyncratic or non-quantitative reasoning; *Transitional*, it begins to recognize the significance of quantitative measures, but frequently there are regressions; *Informal Quantitative*, it involve quantitative reasoning of the form "more of", "less of", "3 out of 5"; *Numerical*, incorporate reasoning with relative numbers and its operations. The authors analyze probabilistic reasoning through six constructs: *sample space*, *experimental probability*, *theoretical probability*, *probability comparisons*, *conditional probability*, and *independence*. Notwithstanding, in this work we regard the fourth ones, because of they are enough to account for the reasoning about the concept of probability of student studied here.

METHODOLOGY

This work is a case study that follows up the development of the reasoning on the concept of probability in one single student, Miguel. He studied the third year of high-school (17 years old) and had a good arithmetic and algebraic reasoning, though he had never studied any probability topics. Only one student was considered in order to explore the underlying ideas between classic

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probability and relative frequency in a deeper way, and his age was determined because, besides having the necessary mathematics knowledge to start working on probability, such ideas have a complexity which is not easy to handle by younger students (Ireland & Watson, 2009, Konold et al. 2011). Fifteen one-hour long interviews were conducted. In six of them the problems in Figure 1 were developed including its simulation using the Fathom software. The problem statements are a variation of a problem taken from Cañizares (1997, p.56). The six sessions were video recorded and transcribed, as well as analyzed based on the framework proposed by Jones et al. (1999).

1. In box A there are 3 black chips and 1 white chip. In box B there are 6 black chips and 2 white ones (fig.1).

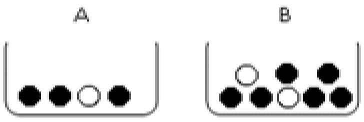

fig. 1

A box is chosen and a chip is drawn randomly from it. A prize is won if it is a black one. Juan considers that choosing box B is the most convenient action because no matter both boxes have a proportional quantity of black and white chips there is a larger number of black chips in B.

a) What do you think about Juan's explanation?
b) How would you measure the possibility that the event "drawing a black chip from box A" occurs? Which value would you assign to it?
c) How would you measure the possibility that the event "drawing a black chip from box B" occurs? Which value would you assign to it?
d) How would you prove the answer given in a)?

2. There are 2 black chips and 2 white ones in box C. Box D contains 4 black chips and 4 white chips (fig. 2).

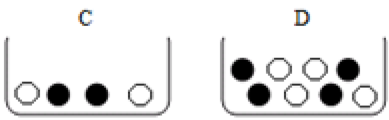

fig. 2

If you had to draw a black chip from one of these boxes to win a prize without looking inside, which box would you choose? Provide a justification to your answer.

3. Two other boxes contain some black chips and some white ones (fig. 3).

Box E: 4 chips in total, including black chips and white chips.
Box F: 2 black chips and 1 white chip.

fig. 3

Which box would you choose to draw from so that the chosen chip is black? Provide a justification to your answer.

Figure 1. Probability comparison problems.

RESULTS

First episode. Miguel wrongly believes that Juan's statement in the problem 1 is right and answers: "I think it is better [to choose box B] because of the [greater] number [of black chips when compared with box A]". Later, he is asked: Which value would you assign to the possibility of drawing a black chip from box A? In his answer, he questions the meaning itself of the assignation: "What do you mean a value?... that's what I don't understand, which value. They are values of what or what…". After a brief intervention of the interviewer, the student shows he perceives the equality in the proportions of black chips: he says there is a "75% of drawing a black [chip from box A]" and a "75% [of drawing a black chip from box B]". However, despite he has apparently calculated the probability of drawing a black chip from each box and observed they are equal, he maintains his appraisal on the situation: "I think Juan is right". As a consequence, this reasoning is placed in a *Subjective* level (Jones et al. 1999).

Second episode. Now, Miguel is asked to draw randomly samples of size 10 with replacement from box A (with one white chip and three black ones) and another one from box B (with two white chips and six black ones), and to register the results in Table 1.

Table 1. Results from the 10 extractions in each box (B = white chips and N = black chips).

Number of drawings	1	2	3	4	5	6	7	8	9	10
Box A	N	N	N	N	N	N	B	B	N	N
Box B	B	N	N	N	B	N	N	N	N	N

Based on these results, Miguel was asked: Which box would you choose if you wanted to get a black chip? He answers he would pick box A and explains: "the first chips drawn turned out to be black". Then, he is asked to draw other samples of size 10, one from box A and another from B. He obtains ten and seven black chips respectively, which strengthens his belief. He is asked now to predict the frequency of white and black chips in 100 and 1,000 possible repetitions of the experiment in each of the boxes (Tables 2 and 3).

Table 2. Prediction of the distribution of 100 extractions.

	Number of white chips	Number of black chips
Box A	25	75
Box B	35	65
Total	60	140

Table 3. Prediction of the distribution of 1,000 extractions.

	Number of white chips	Number of black chips
Box A	250	750
Box B	380	620
Total	630	1370

Miguel's predictions as well as his assignation of values 80% and 70% respectively to the possibility of drawing a black chip from boxes A and B reflect his wrong belief that there are more possibilities of getting a black chip from box A. It is convenient to note that this reasoning is based on a bias called the *law of small numbers* (Tversky & Kahneman, 1982). However, Miguel has set in play the necessary elements that constitute the basic frame of probability: the proportions of chips in the boxes and the relative frequencies, but he has not articulated them yet in a convenient way. Therefore, his reasoning is at a *Transitional* level (Jones et al., 1999).

Third episode. Afterwards, Miguel is asked to represent the situation in the Fathom software and to make the corresponding simulations in which he obtains the following results:

Table 4. Results of the computer simulation of 100 extractions.

	Number of white chips	Number of black chips
Box A	23	77
Box B	29	71
Total	52	148

Table 5. Results of the computer simulation of 1,000 extractions.

	Number of white chips	Number of black chips
Box A	248	752
Box B	246	754
Total	494	1506

When asked what he observes, Miguel focuses his attention in circumstantial characteristics (Table 4): “The more black chips [are in the boxes] […] the less [black chips] are [in the simulation]”. Later on, as a result of the interviewer’s questions on the cases of 100 (Table 4), 1,000 (Table 5) and 10,000 (represented in the software) drawings, Miguel focuses his attention in the similarity that exist between the relative frequency of black and white chips obtained in the simulation and the proportion of black and white chips in the boxes. Besides, he observes that this similarity grows closer as the number of trials increases. Nevertheless, he considers box A offers a greater possibility of drawing a black chip since he reasons with absolute numbers caused he values the differences between results as meaningful: “Black chips come out more often from box A than box B. Here [Table 5] […] there are two [chips] less, but here [Table 4] […] there are six times more”. It’s evident that the Miguel´s observation that box A offers a greater possibility of drawing black chips turned out a relatively strong belief because, in absolute terms, this situation was favored by the results in the physical and computer simulations. Thus, his reasoning remains in the *Transitional* level (Jones et al., 1999).

Fourth episode. Miguel assigns the values 77% and 71% to the possibility of drawing a black chip from boxes A and B respectively, reasserting what happened in the *Third episode*. This assignation is only supported by the results of 100 drawings (Table 4), so the interviewer questions the student so that he assesses how appropriate these values are to the other cases of 1,000 (Table 5) and 10,000 drawings. As a consequence, Miguel integrates two observations he had made during the *Third episode* into his reasoning; on one hand, that the relative frequency of black and white chips obtained through simulation is similar to the proportion of black and white chips in the box; on the other hand, that this similarity grows closer as the number of trials increases: “it is almost a quarter of white chips and three quarters of black chips in 10,000… They are the same, no matter the box [which one choosing]”. He assigns the appropriate values to these possibilities at the same time: “75%”. So, his reasoning has now reached the *Informal Quantitative* level (Jones et al., 1999).

Up to this moment, the articulation of the subjective, frequency and classical interpretations of probability allowed the student to formulate one of the inferences that Stohl and Tarr (2002) point out as important in this process, that is, “the *predictability* in the long-run trends in data (i.e., the law of large numbers)” (p. 321). In order for the student to reach a better understanding of this articulation, i.e., that he grasps the concept of probability, he needs to formulate the inference relative to “the *unpredictability* of random phenomenon in the short-run” (p. 321). In other words, he has to distinguish the role of chance underlying in the outcome of the draw; he has to understand that “selecting the [box] with greater probability of winning does not guarantee a winning draw” (Falk et al., 2012, p. 209).To do so, in a later session, the student is engaged in a second activity, again dealing with comparison of probabilities using boxes (Problem 2, figure 1).

Fifth episode. When solving problem 2 (Figure 1), the student sets in play the knowledge previously acquired: he considers the proportionality between the contents of the boxes to choose which one offers the highest possibilities of drawing a black chip; with respect to the long- run simulation of the experience, he shows an informal understanding of the law of large numbers; and based on this, he assigns the appropriate values to the possibilities involved (50% of drawing a black chip from box C and 50% of drawing a black chip from box D). However, with respect to the short-run simulation, the student shows a misconception, concerning specifically the outcome from the physical simulation of 10 draws from box C and 10 draws from box D (Table 6).

Table 6. Results from the 10 extractions in each box (B = white chips and N = black chips).

Number of drawings	1	2	3	4	5	6	7	8	9	10
Box C	B	B	N	B	B	B	N	B	B	N
Box D	N	N	B	N	N	B	B	B	N	N

When asked which box he would choose for an eleventh draw, if one wanted to get a black chip, Miguel considers that box D offers higher possibilities and claims that “[…] according to the table [Table 6] you get black [chips] more frequently [from box D].” When offered two scenarios in which only two and five draws have been done (Table 6), the student privileges again the experimental outcome above any other previously acquired knowledge: “I would choose in each case the chip [color] you got more times.” This shows that Miguel has not yet articulated the uncertainty of individual outcomes with long-run regularity of the relative frequencies (Metz, 1998), and so he is still in the *Informal Quantitative* reasoning level (Jones et al., 1999).

Sixth episode. When addressing problem 3 (Figure 1), based on an informal understanding of the law of large numbers, the student determines adequately the distribution of black chips and white chips inside box E (1 and 3, respectively) from the outcome obtained in the long-run simulation (10,000 draws); inversely, he uses this informal understanding, to predict an adequate distribution of the outcome if 10,000 draws were done in box F (Table 7).

Table 7. Prediction of the distribution of 10,000 extractions in Box F.

	White chips	Black chips
Box F	3400	6600

Based on his prediction, Miguel was asked which the color of the chip would be in draw 10,001. He answered he would get a black chip based on the absolute frequency with which this result is apparently presented: “there are more black chips [in Table 7].” Also, he assigns a 2/3 value to the possibility that this outcome occurs. In this way, the ability of the student to assign the relative values to which the frequencies tend is clear, but he cannot see the role those values have in inferences in the short (*Fifth episode*) or long-run. In both scenarios, he privileges experimental evidence. So, Miguel is still reasoning in an *Informal Quantitative* level (Jones et al., 1999).

Seventh episode. Finally, the student’s performance is evaluated by means of 10 problems in which he is required to estimate the possibility that both simple and compound events occur. Generally, his performance is appropriate; he only commits mistakes because of lack of attention. In an implicit way, he uses the classical definition of probability in the majority of the problems, and so the interviewer decides to partially institutionalize this concept. To do so, he takes into account the 2/3 value assigned by the student to the possibility of drawing a black chip from a box containing two black chips and a white chip (box F, *Sixth episode*). Basically, the student is explained that what he used to solve the problems is known as the classical interpretation of probability: favorable outcomes divided by the total possible outcomes.

Subsequently, the interviewer returns to what occurred in the *Fifth episode*, in particular to the part related to the physical simulation of 10 draws from box C (2 white chips and 2 black chips) and 10 draws from box D (4 white chips and 4 black chips), in order to clarify the student’s reasoning in short-run simulations. The results of the simulation are shown in Table 6, and based on absolute frequencies, the student considered that box D offered higher possibilities of drawing a black chip: “[…] according to the table you get black [chips] more frequently [from box D].” It seems that the student was able to overcome this misconception through the partial institutionalization of the classical definition of probability: “if I represented it with fractions [classical definition], it would remain the same, it would be the same […]. I would have the same possibility [2/4 and 4/8] of getting a black chip [from boxes C and D, respectively].” Likewise, for box E (*Sixth episode*) Miguel points out the equality between the probability of obtaining a black chip in the first draw and the probability of getting the same color of chip in draw 10,001.

In summary, both the change of representation to express the value to which the frequencies tend and the partial institutionalization of the classical notion of probability allowed the student to articulate the uncertainty of the individual outcomes with the regularities of the relative frequencies that arise from many repetitions of the experience; all of this coming from an informal understanding of the law of large numbers. In this way, the student managed to distinguish the role of chance underlying in the outcome of every draw (Falk et al., 2012) and to see that the numeric value that measures this chance is a solid premise to formulate inferences in situations under uncertainty. Therefore, Miguel's understanding of the concept of probability is adequate, thus reaching the *Numeric* reasoning level (Jones et al., 1999).

CONCLUSIONS

During the interviews, Miguel's reasoning regarding probability follows the trajectory that goes from a *Subjective* level of reasoning to a *Numeric* level (Jones, et al., 1999). In this trajectory, the main inferences the student formulated on the concept of probability, which allowed him to transit through these levels, are consistent with those proposed by Stohl and Tarr (2002), namely: "the *unpredictability* of random phenomenon in the short-run [and] the *predictability* in the long-run trends in data (i.e., the law of large numbers)" (p. 321). Besides, the articulation of these inferences led the student towards the conceptual integration underlying in the randomness construct (Metz, 1998). Nevertheless, he faced a number of difficulties at different moments: overcoming the law of small numbers, handling the variation in a convenient way, giving meaning to the quantification of the propensity of occurrence of an event, articulating the uncertainty of the individual outcomes with long-run regularity of the relative frequencies, and using probability as a premise to formulate inferences. It was observed that three important elements to overcome these difficulties were: an informal knowledge of the law of large numbers, the reasoning with relative quantities and the partial institutionalization of the classical interpretation of probability when it was already used in an informal way by the student.

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