STEP-BY-STEP ACTIVITIES IN THE CLASSROOM PREPARING TO TEACH THE FREQUENTIST DEFINITION OF PROBABILITY

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Due to the explicit insertion of Statistics and Probability into the National Mathematics Curriculum in Brazil, since the end of the twentieth century, increasing numbers of local teachers at the basic school (pre-university) level have shown an interest in learning statistics. This paper describes the details of an activity for explaining the frequentist definition of probability often offered by the author as a workshop in pre-service training courses. Hidden concepts used to facilitate comprehension by teachers in training are presented, and some of the main difficulties encountered by students during the activity are described. These difficulties include the practice of transforming qualitative results (e.g., Heads or Tails in a coin throw) into quantitative ones (i.e., 1 or 0, respectively); the concept of cumulative frequencies; and, finally, the alternation style of Heads and Tails to consider the experiment a random phenomenon.

INTRODUCTION

The basic school division in Brazil, which includes all pre-university education, is established as follows: Elementary school I (5 years), Elementary school II (4 years), and High School (3 years), totaling 12 years of basic schooling. The National Curricula Parameters (NCPs), edited by the Brazilian Ministry of Education (MEC) in the late twentieth century (1997 and 2000) and early twenty-first century (2007), offer guidance to basic education teachers in various disciplines taught during the regular cycle. In particular, documented guidelines are provided concerning the gradual leveling of mathematics curricula for the Elementary and High School levels.

Statistics and Probability are concentrated in a block called Information Theory. Some phrases related to guidance in the topic of Probability that appear in the reference texts include the following: a)"Recognize the random character of natural phenomena and events—scientific, technological, or social—including the meaning and importance of probability as a means to predict (future) outcomes"; b)"By observing the frequency of occurrence of an event and a reasonable number of trials, it is possible to develop some notions of probability." c)"Construction of the sample space and the representation of the possibility of occurrence of an event by a suitable ratio."

Item a) summarizes the tools and abilities that the student is expected to use and gain, respectively, to deal with the topic of probability. Item b) assumes that the probability can be determined from a "reasonable" number of trials, which suggests the frequentist approach for probability. Finally, item c) suggests the use of the classical definition of probability, through the construction of a sample space.

The next sections discuss how probability is presented in textbooks and present a step-by-step activity, developed in the form of a workshop, for both teachers in training and students. Some hints are offered to help the teacher go deeper, and some difficulties that students may encounter are discussed.

BRAZILIAN TEXTBOOKS FOR BASIC SCHOOL

Lima (2001) analyzed Brazilian didactic textbooks for the High School level, elucidating problems concerning the treatment of probability. These problems include, for example, the fact that probability is treated only as manipulative, is not used for decision making, and is not explicitly linked with statistics; students do not consider problems in the textbook to be "meaningful"; and students see the chapter on probability as "boring". Lima (2001) also indicates that the chapter on Combinatorics always precedes the chapters on Probability and Statistics. This order is explicitly suggested by the documents of the MEC, which adopts the trilogy "Combinatorial Analysis, Probability, and Statistics" in sequence.

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In the author's view, despite the recommendation in the NCPs, and although it is an important area, Combinatorial Analysis should be part of the area of Numbers and should not be considered as a prerequisite for the initial learning of Probability and Statistics. Guided by the vision promoted by the NCPs, teachers in training and students have developed a belief that without first developing an understanding of Combinatorial Analysis, nothing about Probability and Statistics can be learned. However, the pre-K12 GAISE Report published by the American Statistical Association (Franklin et al., 2007) mentions that "Counting rules… should be left to areas of discrete mathematics and/or calculus". This opinion was made when discussing the role of Probability in Statistics even for C-level (advanced) learning.

In 2012, together with a group of teachers, the author helped analyze Brazilian textbooks for the High School level, and observed that many of these textbooks begin the discussion of Probability by teaching sample space with equally likely elementary events, using the classical definition of probability by the ratio (number of favorable events/number of possible events). Many authors adopt the use of the frequentist setting; they calculate percentages (relative frequencies) from a table, using data collected on particular observed phenomena, and state that probabilities can be obtained directly from these percentages. To quote from one textbook, "often, relative frequencies are taken as probabilities".

Although item b) in the official document suggests that probability can be obtained by observing the frequency of occurrence, this idea is not always absorbed by teachers, who may have questions regarding the frequentist definition of probability—in terms of its meaning, how to present the idea to students, and the conditions under which it can be considered. To explain the role of relative frequency in defining probability, a step-by-step activity is offered that teachers can use in their classrooms at the Elementary II or High School level. Some theoretical notions behind the steps are presented, to strengthen the teacher's background.

PROBABILITY – STEP-BY-STEP ACTIVITY

Note: words underscored correspond to the *teachers' voice*. Comments (not underscored) are given as suggestions to the teachers.

Start of the Activity

Take a coin of your own – examine it well and state whether it is "honest". The teacher and students together discuss ways to answer this problem. The following activity proposes one potential approach.

Description of Activity

Step 1 Students should group by pairs and take a coin. Consider both faces – HEADS (H) and TAILS (T). Suppose we are interested in whether this coin is "honest". This term means that the chance of obtaining Heads is the same as the chance of obtaining Tails, in the experiment with only these two results.

Step 2 One member of the group should flip the coin, while the other marks the results on the worksheet (Table 3), following these instructions:

a) Draw the coin once and mark line 2 of the worksheet with H or T;

b) Repeat this procedure 30 times, completing all 30 of the cells on line 2.

Note: Table 1 gives an example of a sequence of six possible throws, including all steps from Step 2 to Step 4.

Step 3 Switch places with your partner. Continuing with the spreadsheet (Table 3), go back to items a) and b) above (Step 2), and continue for 30 more throws, for a total of 60 throws (line 2) on the same sheet.

Step 4 Switch places again with your partner. Then:

a) After registering all of the results H and T on line 2, go to line 3. Write 1 for H and 0 for T and complete the line (see example in Table 1). Each member of the group should record half of the results – one partner from n = 1 to 30, and the other partner from n = 31 to 60, always using the same spreadsheet.

b) Next, fill line 4 of the spreadsheet (Table 3). In each cell, place the cumulative number of Heads, until the value of n is reached, where n is the number of throws on line 1. Discuss with another member of the group to see if it is clear - if not, ask the teacher! An example of this line can also be seen in Table 1.

c) Finally, on line 5 of the spreadsheet (Table 3), calculate the relative frequency (**m/n**) of Heads in each cell (use decimals). What is the relative frequency? Discuss this concept with another member of the group. Disregard the entries marked with X – these are used to decrease the time of the activity and do not concern (please use compromise here) the objective of the exercise.

Table 1. Example of a full spreadsheet with six throws of a coin

1) Number of throws (**n**)	**1**	**2**	**3**	**4**	**5**	**6**
2) H or T	H	T	H	H	T	H
3) 1 or 0	1	0	1	1	0	1
4) Heads accumulated (**m**)	**1**	**1**	**2**	**3**	**3**	**4**
5) Relative freq**. m/n**	1/1 = 1	½ = 0.50	2/3 = 0.67	¾ = 0.75	3/5 = 0.60	4/6 = 0.67

Step 5 After completing Table 3, build Table 2, using lines 1 and 5 of the spreadsheet.

Table 2. Trials and cumulative relative frequencies

n	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60
m/n															

Step 6 Complete graph (Figure 1), using values of Table 2 as follows:
Ordinate values → **m/n**
Abscissa values → **n**

Step 7 Compare your results with the results of other groups, and make a global comment about the "honesty" of the coins.

Note. For the last cell (corresponding to n = 60), the expected results are generally between 0.40 and 0.60; however, more extreme values can exceptionally occur. The class can make a table with all relative frequency results for 60 throws, and discuss them with the teacher.

Table 3. Spreadsheet

1) Thrown **n**	1	2	3	4	5	6	7	8	9	10	11	12		14			17			20					25					30
2) H or T																														
3) 1 or 0																														
4) Heads accumulated **m**																														
5) Relative freq. **m/n**											X	X	X	X	X	X	X	X	X		X	X	X	X	X	X	X	X	X	

1) Thrown **n**	31	32	33	34						40							47			50					55					60
2) H or T																														
3) 1 or 0																														
4) Heads accumulated **m**																														
5) Relative freq. **m/n**	X	X	X	X	X	X	X	X	X		X	X	X	X	X	X	X	X	X		X	X	X	X	X	X	X	X	X	

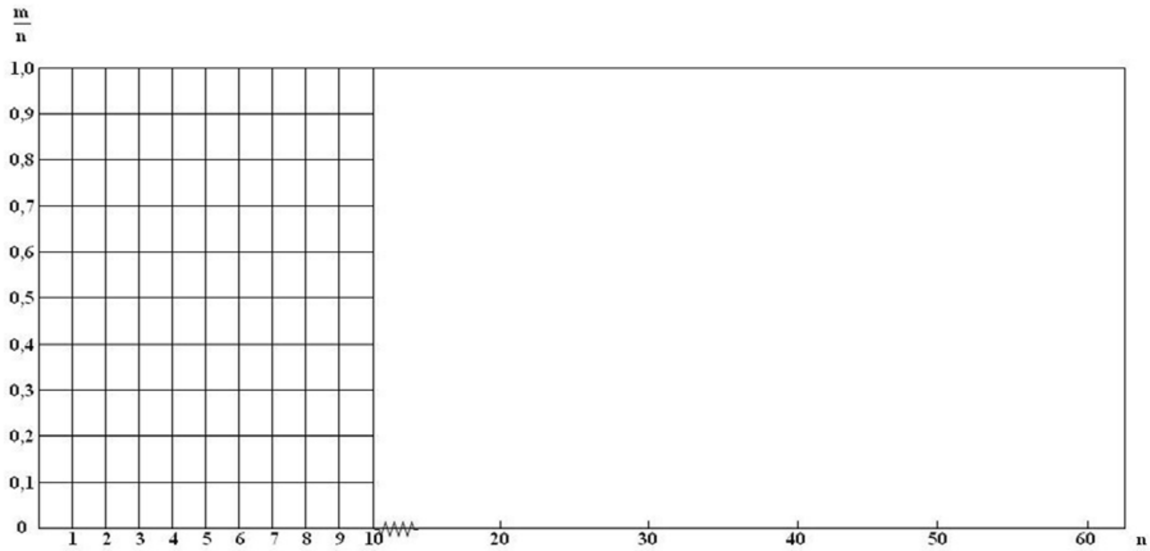


Figure 1. The cumulative relative frequency of times that a Head occurs for 60 trials. ***Note.*** The foreshortening of the horizontal axis is only for convenience here.

PROBABILITY – FREQUENTIST DEFINITION

At the end of this activity, it seems plausible to establish a "frequentist definition" of probability (which has links with the Law of Large Numbers). This definition is as follows: *Probability is the value at which the relative frequency stabilizes after a very large number of trials.*

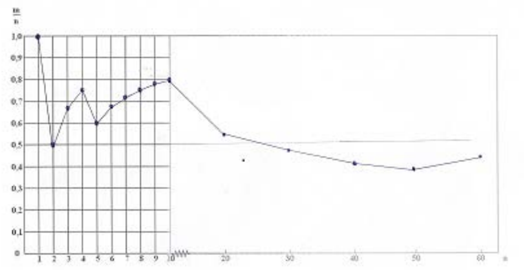
The same approach described here for giving the frequentist definition was used by Jessica Utts (1999), in the chapter "The relative frequency interpretation": "…it makes sense to discuss the probability that the coin lands Heads up. It is simply the relative frequency, over the long run, with which the coin lands Heads up." Agresti and Franklin (2007) stated "In practice: the sample proportion estimates the actual probability" and they suggest the use of an applet for coin flipping, in order to estimate the probability of a balanced coin. This paper offers an alternative with pencil and paper for this activity.

HELPING TEACHERS GO FURTHER

The teacher should keep in mind some theoretical concepts, even if there is not time to work on them in class. These points can be gradually introduced, depending on the students' development and interest. Considering the shown steps, the following aspects can be observed:

a) The random variable "number of Heads in one throw" (Y) takes a value of 1 (Head) or 0 (Tail). This variable can be associated with the Bernoulli distribution with parameter **p**, which is the expected value (mean) of Y and the probability of Heads for the coin under investigation. Therefore, what was sought was the value of **p**, that is, P(Head);
b) **m** = $\sum_{i=1}^{n} Y_i$ (where each Y is a Bernoulli, as mentioned in a);

c) The plotted values in the graph (Figure 1) present a greater initial variability, which decreases with n. The final behavior seems to reach some kind of stability. This aspect encourages the development of a frequentist definition of probability; that is, **p** = P(Head) = $\lim_{n\to\infty}\left(\frac{m}{n}\right)$. Shown below is a sequence of 60 throws of a unbiased coin and the corresponding graph, which could be obtained through the steps above:



1 0 1 1 0 1 1 1 1 1 0 0 0 1 1 0 1 0
0 0 1 0 0 0 0 0 0 1 0 1 0 0 0 1 1 0
0 0 0 0 1 0 0 0 0 1 0 0 1 0 1 1 0 0
1 0 1 0 1 1

d) The obtained value for the last cell of line 5 (Table 3) corresponding to n = 60 is the relative frequency of Heads after the 60th throw ([**25/60**] **=0.42**);
e) From one experiment of this type, it is possible to discuss the frequentist definition of probability for tables with relative frequencies, taking into account, for example, the number of observations (**n**) to obtain the required stability. This context also emphasizes the fact that probability relates to a future event.

STUDENT-PERCEIVED DIFFICULTIES IN THE ACTIVITY

This activity involves several steps. For each one, time for reflection is necessary. Once the algorithm is understood, filling in the table is automatic. However, students may encounter obstacles at the beginning of the activity that could produce some insecurity for teachers. Next, this paper highlights three types of difficulties encountered with this activity during the dozens of training workshops taught by the author since 2000, in different parts of Brazil:

a) Difficulty in understanding the transition from a qualitative representation (*H*, *T*) to a quantitative one (1, 0): The first representation takes into account the question, "What is the result of the coin toss?" The answer is a category (Head or Tail). The second representation is quantitative and relates to the question, "What is the number of Heads on a toss of a coin?" The possible answers are one and zero. Line 2 of the spreadsheet refers to the qualitative aspect and line 3 to the quantitative one. This last approach makes it easier to achieve the required frequencies, by counting the "ones". Although the choice is arbitrary, it is convenient to represent the category of interest of the problem by one (1), for situations that permit the use of the Bernoulli distribution approach.
b) Difficulty in filling in line 4 (the cumulative frequencies): Working in pairs facilitates the exchange of ideas, but the teacher should be aware that sometimes the duo cannot start or does not complete the line properly. In particular:
    - There is a certain delay needed for the students to absorb the concept of the cumulative frequency of Heads until a specific toss n. Once the process is understood, the filling of the spreadsheet follows almost automatically;
    - There is a tendency not to make an entry for the corresponding repetitions of Tails on line 4. The student should repeat the previous total of Heads until the new result of Heads (if any). The teacher should be aware that some pairs leave the cells corresponding to Tails empty, which can make completing line 5 difficult.
c) Surprise with long repeated runs: The student may consider three or more repetitions of Heads (or Tails) as sufficient to suspect that the coin is biased. They think the best behavior is a systematic result (e.g., H T H T H T H T). The teacher should make a comment about this point, noting that systematic behavior is a non-random sequence. The simulation presented in p. 3 was generated by a (*pseudo*) random process. It is easy to see three sequences of five runs, and other sequences as well:

1 0 1 1 0 1 1 1 1 1 0 0 0 1 1 0 1 0 0 0 1 0 0 0 0 0 0 1 0 1 0 0 0 1 1 0 0 0 0 0
1 0 0 0 0 1 0 0 1 0 1 1 0 0 1 0 1 0 1 1

Note. The teacher should clarify that in 10 tosses of a non-biased coin, any systematic balanced sequence such as HTHTHTHT, or THTHTHTH, or THHTTHHTTHHT or HTTHHTTHHTTH has a very low probability of occurring in a random generation scheme. The students can be invited to see how many of them can find such a sequence anywhere in the lists of outcomes.

FINAL COMMENTS

From our perspective, step-by-step activities facilitate the construction of some concepts, allowing time for reflection and comprehension, both during the task and afterwards. These aspects were illustrated in this activity, in which the concept of probability was introduced by means of the limit of relative frequencies. For teachers in training, emphasis should also be given to hidden concepts that, although not obvious at first to students at this moment, are present in the development of the task. Other items of importance are the difficulties that students have in developing some steps, which can be addressed during the course of the activity, to achieve the final result. Other topics of Probability and Statistics can also benefit from this strategy, to support the understanding of the area. This approach can be used not only to modify the classroom expositive style to a proactive one, but could also be included in Mathematics textbooks for the pre-university level.

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