A STUDY ON MATHEMATICAL ACTIVITY WITH A FOCUS ON LINKING THE THEORETICAL AND EXPERIMENTAL APPROACHES TO PROBABILITY

Takuma Takayama
Shimoda Junior High School, Tokyo, Japan
takumata@gmail.com

We propose mathematical activity to obtain statistical probability by experiment. Students can connect probability and statistics by this activity. Furthermore, this activity makes students to think about conditions by which statistical probability converge to mathematical probability. Furthermore, with this activity, students can acquire the law of large numbers, and also notice the idea of statistical testing.

INTRODUCTION

Since 17th century, Japanese have played a game called "Janken," which is worldly known as "Rock-Paper-Scissors." In Japanese mathematics textbooks, Rock-Paper-Scissors as well as "Coin Toss" are often used for typical probability problems. In "A Fair Game? The case of Rock-Paper-Scissors," National Council of Teachers of Mathematics (NCTM) also reported an experiment of Rock-Paper-Scissors with three players.

The aim of this research is to make students realize how to connect probability and statistics through experiments in which we adopt an activity to find statistical probability. We make them be aware of the following three points as well.

1. According to the law of large numbers, empirical probability converges to statistical probability.
2. Even though the way to choose a hand-sign is biased, the distribution of the possibilities of a draw becomes a binomial distribution B(n,p) and the central limit theorem guarantees that it comes close to a normal distribution N(np, np(1-p)) as the sample size n becomes large.
3. When the elementary events are not equally possible, the statistical probability is not necessarily equal to the mathematical one.

The outline of the lesson plan is as follows:

1. Make a prediction
2. Find a statistical probability based on the result of experiments
3. Share the statistical probabilities of each group in the whole class and combine them.
4. Calculate the mathematical probability by using a tree diagram and verify the result of the experiments.
5. Have a discussion (In some cases, the procedure will be ①⇒④⇒②⇒③)

RELATED WORKS AND FRAMEWORK OF THIS RESEARCH

Most of the previous studies of how students acquire the concept of empirical and mathematical probabilities assume that students regard mathematical probability as the true one. On the other hand, Konold *et al.* examined problems which the student confront with when they try to make sense the consistency between the empirical probability and the mathematical probability. In it, Konold *et al.* pointed out the problem of these previous studies tacitly assuming that students regard the mathematical probability as the true one.

In line with the textbook for the junior high school students in japan, this research defines the limit of a relative frequency obtained by an experiment as statiscical probability. Since elementary events are equally possible, the mathematical probability is defined as follows: (the number of outcomes of an event) ÷ (the number of all possible outcomes of an experiment) =(mathematical probability). The aim of this research is to investigate whether students are able to realize the following four points or not by the analysis of the students' worksheets and the dialogues with them.

① The empirical probability is not necessarily equal to the mathematical probability.

② When the elementary events are equally possible, the relative frequency converges to the mathematical probability as the number of trial becomes large.

③ When three persons play Rock-Paper-Scissors, the binomial distribution of the probability of a draw comes close to a normal distribution as the number of trials increases (the binomial distribution has not been taught yet).

STUDENT ACTIVITIES

In modern times with high uncertainty the need to read trends and to make judgments by using statistics are increasing for making a decision. It is therefore very important to have an ability of accurately understanding the meanings of both statistical and mathematical probabilities and an ability of judging the likelihood of events.

In the second grade of junior high school, students learn that statistical probability is equal to mathematical probability when elementary events are equally likely. After that, they acquire basic calculation skills of probability. A complete understanding of the relationship between statistical probability and mathematical probability or an understanding of events of "equally possible" will be helpful for their studies in high school.

In this lecture, when students come to be able to find the mathematical probability by using a tree diagram, we return to relative frequency and make the students think over the condition on which the relative frequency converges to mathematical probability in actual phenomena. For example, when four people play Rock-Paper-Scissors, the rate of a draw unfortunately does not always converges to the mathematical probability, 13/27. In this situation, if the students are asked why it does not converge to the 13/27, their experiences such as "there is a person who always shows rock" or "there are many people who show paper in the first round" lead them to realize that the elementary events of Rock-Paper-Scissors are not equally possible. Furthermore, there are some concepts of probability students have already acquired by their daily experience. For instance, they know it is more likely to end in a draw as the number of player increases, but they do not know the mathematical reason behind it. The mathematical activity in which students actually calculate mathematically the probability by themselves enables them to reconfirm that their knowledge acquired during their experience is correct and that they come to be able to explain the mathematical reason for it. This is another aim of this activity.

In addition, we make the students think over how to make the statistical probability of Rock-Paper-Scissors be equal to the mathematical probability. As mentioned above, the elementary events of Rock-Paper-Scissors are not equally likely events. We make them realize that using cards to choose the hand sign for Rock-Paper-Scissors would be a solution.

DETAIL OF THE PARTICIPANTS AND THE ACTIVITIES

The following question is on the textbook.

When two 10 yen coins are tossed, which case would be most likely? A. Both are head. B. One is head, and the other is tail. C. Both are tail.

① Make a prediction

At first, make the students predict the probability of A, B, C. Most students can expect that the probability of A and that of C are the same. It is difficult for them to expect the probability of B.

②Let's conduct an experiment to test it.

Make the students think over a method of the experiment.

・ How do we toss two 10 yen coins?
・ How can we efficiently collect more data?

Respect the students' opinions while conducting experiments, which helps their active learning.
In this lecture, the following opinions were voiced.
Put the two 10 yen coins in a paper cup, and shake the cup, then check the outcome.

- Make pairs. One tosses the coins, and the other records the outcomes.
- One of the pairs records the 50 outcomes, and counts the number of the events of head-tail. Then, writes down the result on a blackboard.

③Sharing data

In order to share the data, a student suggested making a graph. We decided to draw a horizontal line with a scale of 0 to 50 and to mark with a circle by chalk. We made pairs and each pair tossed a coin fifty times, and all the data obtained by fifteen pairs were put together. It was confirmed that the relative frequency of the events B was almost 0.5.

④Make the students calculate the mathematical probability.

I asked, “Is the probability always 0.25?” and made them think it over in a group. I said, “Let’s make a group of four,” and I distributed a whiteboard for their presentation to each group and said, “find the probability of ‘one head and one tail’ by using a tree diagram.” We moved to a group working.

The next question is the probability of Rock-Paper-Scissors. Make the students think over the following question.

When three people A, B, C play one round of Rock-Paper-Scissors,
(1) What is the probability that only A wins?
(2) What is the probability that only A loses?
(3) What is the probability that three of them draw?

The students found the mathematical probability by using a tree diagram. After solving this problem, ask the following question to the students.

When two people play Rock-Paper-Scissors, the probability of a draw is 1/3. When three people play it, the probability of a draw is 1/3. When the number of players becomes 4 and 5, are the probabilities of a draw 1/3?

The students’ opinions:
In this class, the following opinions were voiced.
“When I play Rock-Paper-Scissors with 4 players, it is more often to end in a draw. So, I don’t think that the probability of a draw is always 1/3.”
“I agree with you. That’s why we have a special rule called ‘oi-gachi’.”
< How do we confirm the supposition? >

A student said, “A experiment is a good idea to test the supposition.” Therefore, we decided to conduct an experiment. I said, “We found the number of outcomes of throwing a one with one dice by conducting a experiment of a dice roll. Let’s discuss in your group what kind of experiment we should do for finding the probability of a draw.” The following opinions were voiced:
“Three people play ten rounds of Rock-Paper-Scissors and count the number of outcomes of a draw.”
“Although it is difficult to play 1000 rounds of Rock-Paper-Scissors, we can divide it among the whole class.”
An opinion such as “How about replacing the group halfway through the experiment?” was voiced as well.

< How do we conduct the experiment? >

Since the students have already conducted experiments of probability several times by that time, the following ideas were naturally suggested.
① Make a group of four, and play thirty rounds of Rock-Paper-Scissors.
② Write down the data on the blackboard in order to share them.
③ Add up the data and find the empirical probability of a draw.
We made three groups of four, and played thirty rounds of Rock-Paper-Scissors to calculate the empirical probability.
The probabilities of the groups were 17/30, 18/30, 14/30, respectively. Therefore, the sum of them became 49/90.

FINDINGS INCLUDING STUDENTS’ RESPONSES RELEVANT TO THE TOPIC

< Discussion about the way to find the mathematical probability >

When I asked how to calculate the mathematical probability, a student answered, “How about drawing a tree diagram?”
I handed over a whiteboard to each group, and put it on the center of each group. I asked them to list all the possible outcomes without duplication and to draw a tree diagram. By using the tree diagram, they were able to find the probability, 17/27.

< Discussion about whether the mathematical probability becomes equal to the relative frequency >
T: Will the experimental probability of a draw really be 17/27?
S: If you repeat the experiment many times, I think the probability will become 17/27.
S: Unlike the experiment of the coin toss, I think that the empirical probability will not be equal to the mathematical one, because some players are strong in Rock-Paper-Scissors and others are weak.
T: So, how can we make the probability 17/27?
S: It is better to change the opponents every time you play it in order that the rate of choosing hand signs is equal to one another.
T: Why do you think so?
S: Because the way to choose the hand sign would be fair.

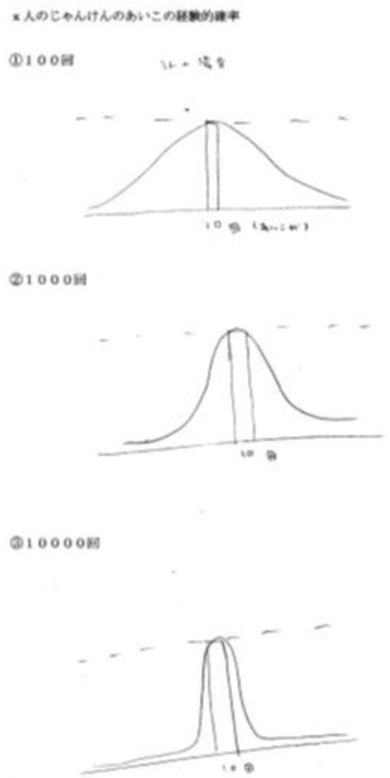
Based on his experience, this student understands that increasing the number of experiment has no effect for the empirical probability to converge to the mathematical probability if the way to choose a hand sign is not random and if the elementary events are not equally possible. Furthermore, he understands that the empirical probability converges to the mathematical probability if you choose hand sign by using coin toss or cards.

< Is there any other method to find the probability? >
When two people play Rock-Paper-Scissors, we can find the probability by making use of the mutually exclusive events. I want to make the students recall it and think over how to find the probability of Rock-Paper-Scissors among three players. The students were able to realize logically that the match of Rock-Paper-Scissors comes to an end only when the number of the types of hand signs all the players showed is two.

< Discussion about the convergence of the empirical probability >
I asked, “When three people play Rock-Paper-Scissors, the mathematical probability of a draw is 1/3.
Conduct a set of experiment in which three people play 30 rounds of Rock-Paper-Scissors, 100 times, 1000 times and 10000 times and plot the results of them on three graphs with the number of draw in 30 rounds as the horizontal axis. How does the graphs look like?”

Most students were able to draw figures like this (Fig.1). By conducting the experiment, they understood it by intuition that the probability converges to 1/3 as the number of trials becomes large. This means that from their daily life they have empirically known that the probability becomes a binomial distribution B (n, 1/3) and the distribution comes close to a normal distribution N((1/3)n, (2/9)n) as n becomes large. Their conversation in the classroom and the worksheets written by them indicate that they realize the fact that the empirical probability comes close to the mathematical probability and the number of a draw in a set of experiment comes close to the mathematical probability as the number of experiments increases.

Fig.1 Student Answer Sheet



DISCUSSION AND CONCLUSION

This research revealed the following five points.

(1) By the experiment of Rock-Paper-Scissors, confirming the way the empirical probability converges to the mathematical one brings the students an opportunity to think over the meaning of probability and the law of large numbers.

(2) In the process of consideration about the experiment of Rock-Paper-Scissors, they grasp the concept of the elementary events and realize that we need to use something like cards for choosing a hand sign in order to accurately calculate the probability.

(3) Even though the way to choose a hand sign depends on each player and is not random, the central limit theorem assures that the binomial distribution of the number of a draw in 30 rounds becomes a normal distribution as the number of experiments becomes large. The students had understood it by the experiment and by their daily-life experience.

(4) Though a junior high school is the first place for the students to learn the probability, it is important for them to make sense their knowledge acquired during their daily life by building a theory based on the consideration of experiments.

(5) In general students learn mathematical probability after they have learned statistical probability, however, it is helpful for them to deeply understand an elementary event that they conduct experiments and think over statistical probability again after they learned mathematical probability.

FINAL REMARKS

Adopting an experiment into the probability class provides a good opportunity to consciously connect the probability with statistics. Instead of taking it for granted that the probabilities of elementary events are equal to one another, I would like to develop the students' critical thinking such as "is it really true?" Through an active learning, I also expect them to develop their ability to understand that statistical probability converges to mathematical probability by using the law of large numbers.

NOTE

When we play Rock-Paper-Scissors with many players, we adopt 'oi-gachi' rule, in which the hand sign the most players showed wins.

Make the students notice that they empirically learned the "oi-gachi" rule and have already practiced it.

(When we play Rock-Paper-Scissors with many players, we adopt 'oi-gachi' rule, in which the hand sign the most players showed wins.)

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