UNDERSTANDING RANDOMNESS: FROM GIMMICKS TO EFFECTIVE LEARNING

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Activities for students that highlight important probabilistic ideas may appeal to teachers because they offer the chance to do something "fun" in the classroom, in contrast to "routine" activities. There is, however, a risk that these activities are not taken beyond the "gimmick", falling short of providing deep and effective learning of key principles. When teacher educators share such an activity with teachers, how do they help teachers know how to use the activity effectively in the classroom, to go beyond superficial "entertainment" to deep "education"? Several scenarios, including an activity about coin-tossing runs intended to highlight features of randomness, will be used as examples to discuss issues such as identifying key concepts, using simulations effectively, determining "how much is enough", and connecting theory with experiment.

INTRODUCTION

I have a little party trick that I conduct with my pre-service teachers when we are doing our sessions on teaching probability (unfortunately, I cannot remember the source of inspiration for this activity). I ask them each to toss a coin, thus randomly allocating them to two groups based on the outcome of this toss. Those in the tails group are now asked to toss the coin 50 times, and record the sequence of tosses, e.g., HHTHTHTTTT… . Those in the heads group, in contrast, are asked to *pretend* that they are tossing a coin 50 times, and thus fake what they think is a convincing sequence of 50 coin tosses. They are thus to write down a list of heads and tails that would make me think they had really tossed the coin. After giving these instructions, I leave the room, and allow students to generate their sequences. In fact, I ask the students to repeat the process (randomly allocate themselves to a group, and then follow the real-generator or fake-generator instructions), so that each student has two sequences prepared by the time I return. I proceed to examine all the sequences and adjudge them as real or faked. My students are often quite impressed with my performance on this, because I have about a 75% success rate for correctly guessing whether or not the student generated the sequence by actually tossing the coin or by faking the process.

The question is, what happens next? What do I do, as a teacher educator, with this experience? What do I intend for my pre-service teachers to learn from it? What will they end up doing with it? Will they use it themselves in a classroom, and how?

Before addressing these questions, here is another scenario. Consider a high school teacher, who knows about the birthday "paradox" and who wants to use it in a lesson on probability because she thinks it will be engaging. She realises that there are 24 students in her class—with no twins to "ruin" the scenario—and so she asks the students to list their birthdays. They do so, and eventually one of two things happens: she either gets a match … or she does not. What happens next? What does she expect her students to learn from it?

SOME CLASSIC PROBABILITY ACTIVITIES

"Randomness" is a central idea for probability, so gaining an understanding of it is essential for learners. However, the concept is complex and the effective teaching of it is challenging. Comprehending randomness requires integrating multiple phenomena such as a long-term constant behaviour (likelihood), and short-term uncertainty (variation)—that is, both predictability and unpredictability. The very non-deterministic characteristics that define it are what make it difficult to structure teaching activities that convey the central ideas, since, for example, it requires more than an experiment that reveals 31 tails out of 50 tosses to make a case for a fair coin having a 50% probability of coming up heads.

There are a number of "standard activities" that have become part of the repertoire for teaching probability. They are often passed on as a kind of "folk-lore", shared during professional learning sessions or from teacher to teacher, or perhaps they become a little more formally

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"authorised" when presented as activities in textbooks. Here are some examples (not all of these may be as "standard" or as well-known as suggested, but they serve to illustrate some important points later; note, too, that these may be variants of activities that you may know in different guises):

- *Tossing a six-sided die:* A die is tossed a certain number of times, and the outcomes are tallied and then graphed. Sometimes the data obtained by separate individuals are combined to give a larger number of tosses in total.
- *Horse racing:* Twelve counters, numbered 1 to 12 and representing horses, are placed on a racetrack, a certain number of steps or spaces long. Two dice are rolled and the outcomes summed, and the "horse" with that number then gets to move one space or step forward. The winner is the first horse across the finish line. [As is well known, horse number 7 is more likely to move than any of the other horses, while poor horse number 1 never leaves the start.]
- *The two spinner game:* Two spinners, each equally likely to generate any one of the numbers from 1 to 9, are spun and the results summed. If the total is odd, then player 1 gets a point; if the total is even then player 2 gets a point. The winner is the first player to get 10 points. Is the game fair? [Source: Feely (2003); see also Baker and Chick (2007) and Chick and Baker (2005) for some discussion of this activity.]
- *The birthday "paradox":* Discussed earlier, the phenomenon in question concerns how likely it is that in a certain-sized group of people there are two people who share the same birthday. Surprisingly (hence the attribution of "paradox"), it takes a group size of only 23 for the probability of "birthday twins" to exceed 50%.
- *Real and fake coin tossing sequences:* The success of the leader in the first activity described in the introduction relies on the participants' likely lack of knowledge of the characteristics of coin tossing sequences (this will be discussed in more detail later).
- *"Monty Hall paradox":* The famous Monty Hall paradox arises from a game show in which the host shows the competitor three doors, one of which hides a car, while a goat is behind each of the other two doors. The host—who knows what is behind the doors—invites the competitor to choose a door, and, after hearing the choice, opens one of the other doors to reveal a goat. The competitor is given the opportunity to switch from the chosen door to the remaining closed door, after which the competitor's final choice of door is opened to reveal the prize. Many people argue that there is nothing to be gained by switching choices, since there are now two doors—one with a goat and one with a car—and so, for each, the chances of there being a car behind it is 50%. However, the revelation of the first goat gives, in effect, additional information about the situation, which means that switching choices actually gives you a 2/3 chance of winning the car.
- *What's in the bag?* Described in an article by Brousseau, Brousseau, and Wakefield (2001) after they conducted the activity with a fourth-grade class, this involves trying to figure out the contents of a bag containing five counters, some black and some white. Single counter samples can be taken, with the counter replaced each time.

GIMMICKS AND PRINCIPLES

Each of these activities can be used to illustrate certain principles of probability, as suggested below. In all the cases, there are aspects that can be usefully illuminated by using simulations, which might verify theory, provide evidence for a hypothesis, or allow other insights.

- *Tossing a six-sided die:* This activity is typically done in order to highlight that each of the outcomes occurs about one-sixth of the time, and thus build up the idea of likelihood as a long-term phenomenon. The graph of the combined data is often used to strengthen this claim, since the one-sixth-ness is more noticeable, with the proportional variation across the individual totals smaller than occurs when fewer tosses have been graphed. In addition, however, the activity also allows exploration of variation as a shorter-term phenomenon, to give a sense of what affects how close to or how far from one-sixth the proportions occur.
- *Horse racing:* Consideration of the relatively simple sample space behind this scenario reveals that the different possible outcomes from rolling two dice and adding have different likelihoods,

thus resulting in the realization that some horses are more likely to win than others. Further work might allow exploration of the likelihood of different horses winning races on tracks of differing lengths.

- *The two spinner game:* In theory—and the theoretical exploration of the sample space is an important aspect of this activity—the spinner game is unfair, with Player 2 (the even player) slightly more likely to win than odd. As it happens, with the difference in likelihoods being quite small, an important question to consider is whether or not the unfairness is likely to be observable in practice.
- *The birthday "paradox":* Determining the probability of getting a "twin" in a group of a certain size requires understanding of independent and complementary events. The resultant mathematics reveals how rapidly the likelihood grows as the groups get bigger, with the likelihood greater than 90% by the time there are 41 people in the group. The activity can also stimulate useful discussion of probabilistic scenarios which have counterintuitive results, since the abundance of different birthday dates might suggest that finding "twins" is less likely than actually occurs.
- *Real and fake coin tossing sequences:* This activity allows an exploration of the characteristics of randomly generated sequences, including the proportional occurrences of different outcomes and the likely length of "runs" (a run is a subsequence of the sequence in which all outcomes are the same, such as TTTT).
- *"Monty Hall paradox":* The paradox can be resolved by considering conditional probability, or by representing the situation diagrammatically or with relative absolute frequencies (see the "Monty Hall paradox" entry in *Wikipedia* (https://en.wikipedia.org/wiki/Monty_Hall_problem) for some examples of different approaches).
- *What's in the bag?* This activity—in fact, a series of activities carefully designed by Brousseau et al. (2001)—allows students to explore ideas of randomness, sampling, using probability and statistics to make predictions, and how much uncertainty might be associated with a prediction.

There are, however, risks that the activities are used in ways that reduce them to mere gimmicks, or, at least, which reduce their effectiveness for conveying important principles. Some possibilities are described below:

- *Tossing a six-sided die:* When this activity is used, it is often the case that only the long-term "one-sixth" proportion is highlighted, to the point where some students, asked to draw a "typical" graph for, say, 60 tosses, are happy to have all outcomes occurring exactly 10 times. In this case, perhaps it is true that the idea of equal likelihood and the actual proportion of "one-sixth" has been learned (at least to some extent), but the issue of variation has not. The idea of randomness may thus be only partially addressed.
- *Horse racing:* This activity is sometimes used too early in the curriculum, so that although some of its results become known—e.g., "7 is the best horse"—yet this may have been investigated only at a superficial level. There may be some discussion of how the outcomes are obtained, but perhaps without a comprehension of the full sample space (e.g., it is possible to conclude that "7 is the best horse" without being careful about the fact that 3 on one die and 4 on the second is a different outcome from 4 on the first die and 3 on the other). Moreover, students may assume that because "7 is the best horse" then "7 will win the race most of the time", when, in fact, on a 10-space track 7 only wins about 40% of the time (it is, of course, more likely to win than any other horse, but this is not the same as winning "most" of the time). Sometimes the activity is used simply to show that there are situations where not all outcomes are equally likely, as a foundation for later exploration, which is fine as a learning objective, up to a point. However, students can come to think they "know" the activity and its lessons, having "done it" once and having had only a superficial examination of the situation, and they may not be inclined to engage with it again if the activity is revisited later in the curriculum for more in-depth study.
- *The two spinner game:* As described in the teachers' resource book (Feely, 2003), teachers might expect that this game should appear to be biased in actual play. Unfortunately, with a 41/81 chance of getting even and a 40/81 chance of getting odd, using the spinners to produce points

for "odd" and "even" is not far off being equivalent to tossing a coin to determine odd and even, and even in a "race to 10" there is only a 52% chance that even will win (see Chick, 2010), which is not likely to be visible in routine classroom play. It is a nice game and good for discussing the sample space, but, for showing a biased game, this activity is not as effective as might be hoped. Moreover, incorrect arguments can lead to a correct determination of the bias (see Baker & Chick, 2007; Chick, 2010). Some adaptations for this game which are more suitable for investigating the idea of bias are presented in Baker and Chick (2007). For more experienced students, the game as it stands *is* useful for discussing the impact of small biases, but the teacher has to realize that this can be addressed, and figure out how best to do so.

- *The birthday "paradox":* The scenario described in the introduction of this paper is one way in which I have seen this activity addressed: grab a group of about 23 people and see if a match occurs. There is a difficulty here, though. What do students learn from seeing that their class contains a match? Will students be *over*-convinced about how frequently this occurs, and expect it to happen in *most* groups of 23? On the other hand, what do students learn if the class does not have any "twins"? Will they be *un*-convinced about the relatively high likelihood of getting twins? The fact is, neither outcome gives any sense of a 50% likelihood; a single group as the source of data for the scenario is not sufficient. Ideally, *multiple* groups of 23 need to be tested, if the idea is to give a sense of how likely it is to have a match. If used only with a one-off sample, students *may* remember the "23-people/50%-chance-of-twins fact" but may not have a deep sense of what this truly means, nor know how the result is verified. Callingham and Watson (2004) suggest using the birth-*month* scenario as an alternative, where, instead of looking for a date in common you look for a month, for groups of five people. This can be simulated in classrooms by randomly selecting groups of five students; they also suggest getting students to devise ways of simulating the larger collections of groups of five, before doing the theoretical analysis.
- *Real and fake coin tossing sequences:* It is easy to imagine reducing this activity, as hinted in the introduction, to one in which the "psychic" teacher performs some "magic" and correctly guesses whether the sequences were real or faked. The students may well be impressed, but clearly more work is required to have students appreciate the way in which randomness manifests itself in the properties of the runs that appear in sequences. The teacher is, of course, relying on the fact that people unfamiliar with these properties will find it difficult to write a fake sequence containing runs of four or more consecutive heads (or tails): over 80% of sequences of 50 tosses will have at least one run of length 5 or longer. However, if only used as a gimmicky trick, the full lessons may be not only lost but actually un-taught.
- *"Monty Hall paradox":* Like the other activities discussed here, the lessons that can be learned from the Monty Hall paradox scenario depend on the way in which the scenario is investigated. There is potential to make important connections among the different ways of representing and resolving the problem, and to explore students' own probabilistic reasoning as they propose different explanations for what is occurring. Such conversations are challenging for the teacher to manage, and there is a risk that the teacher provides a solution directly without allowing the rich discussion that is possible here.
- *What's in the bag?* The objective for this activity was to predict the contents of the bag. When Brousseau et al. (2001) implemented the activity, they made a deliberate choice *not* to reveal the contents of the bag after students have made their predictions. Many teachers would find this disconcerting, and would surely be tempted to allow students to validate their predictions by looking inside the bag for the "answer". To do so would omit one of the key points of the activity, that oftentimes statistics (and the probabilistic underpinnings of sampling) are being used to make predictions without knowing the answer; that statistics can predict the likelihood of the answer, and allow for the uncertainty in the prediction.

DISCUSSION

The examples discussed above highlight the complexity of teaching probability—particularly the idea of "randomness"—and give some insight into what is required of teachers in order to do so effectively. The interplay of content knowledge and pedagogical content knowledge (Shulman, 1986) is particularly striking. A deep understanding of randomness and how it is manifest in probabilistic phenomena is clearly essential here, but the teacher must understand how to make this evident in the classroom to ensure these activities move beyond gimmicks to become effective learning activities that convey essential principles to students.

Gibson's 1977 theory of affordances is useful here. He suggests that things—objects, but also teaching materials and activities—hold an inherent potential to be able to be put to use in some way by us, if only we recognize that potential. For example, a lightweight plastic chair can be used as a seat, a bookshelf, and a rain shelter; each of these three things is an affordance of the chair. Class discussion of the birthday paradox affords the opportunity to discuss whether testing the activity on one class is going to tell us much; highlight the role of complementary events, independent events and the multiplication rule; and investigate what happens as the group gets larger. It also affords the opportunity to be presented as a "test just one class" stunt, or to be simplified for accessible analysis as in the birth-month variation suggested by Callingham and Watson (2004).

The list below suggests some key aspects of content knowledge and pedagogical content knowledge that are likely to be necessary for the effective teaching of probability and, especially, randomness.

- *Profound understanding of randomness:* The teacher's own understanding of randomness needs to be deep and secure. Without good understanding, it will be difficult to recognize what concepts can be emphasized with the activities, and it will be too easy to reduce an activity to a mere gimmick or trick.
- *Knowledge of student conceptions:* Awareness of students' typical misconceptions is essential for teachers, together with a repertoire of strategies for addressing them. Helping students to understand whether "luck" exists, why an unbiassed coin that has just come up heads five times in a row is still only 50% likely to come up tails on the next toss, and that 3+4 is different outcome from 4+3 in the horse racing game, are just some of the many issues that will arise for some students and which need to be addressed effectively.
- *Know how the activity illustrates the phenomenon:* Having strong content knowledge makes it possible to recognise the affordances of a particular activity, i.e., to identify exactly what principles can be demonstrated with the activity. In addition to recognising the affordances, however, the teacher must also know exactly what features of the activity make the principles evident and how to emphasize these appropriately. For example, doing the birth-*month* problem (rather than the birth-*day* paradox) allows a simpler exploration of the theory and a more manageable simulation for generating lots of examples of groups of size five.
- *Know how much evidence is needed to demonstrate the phenomenon:* Given the inherent unpredictability of random generators (which is precisely what makes them random!), the teacher needs to be aware that illustrating a phenomenon requires statistical evidence not just "existence" evidence. Getting twins in a single class of 23 people does not help students to see the 50% likelihood of such an event; 14 occurrences of a "5" when rolling a die 60 times is not evidence of a "one-sixth" probability; and having horse "8" win the horse race game (which will happen around 20% of the time on a 10-space track) does not provide evidence for the greater likelihood of "7" … and yet all of these things can occur in an implementation of the activity. Brousseau et al. (2001) made specific comment about how the effectiveness of an activity can be affected by the chance occurrences of outcomes different from what is "needed" to show the principle required.
- *Build bridges from situation to simulation:* For many probability activities, simulation provides a powerful way to get the large-scale repetitions of an activity that generate the required sample sizes for producing evidence for probabilistic phenomena. It is important to provide a "bridge" from the real situation to the simulation, so that students understand how the simulation generates what they were doing in the initial activity. In the case of rolling a die, for example, it is important to indicate what acts as the die in the simulator, discuss how the simulator allows it to be rolled 600 times, explain how the results are tallied, and describe how the graph from the simulator is like the graphs produced earlier from the initial data. In the case of the Monty Hall paradox, for instance, the simulator has to be able to simulate a choice and the opening of a goat door and the option of changing.

- *Recognise the power of simulations:* As suggested above, simulations provide the opportunity to gather enough data to make a phenomenon more evident than might otherwise be the case. This provides impetus for developing a theoretical understanding of a phenomenon. As an example, contrasting 60 simulated rolls of a die, with 600 and then 6000 simulated rolls of a die allows a better understanding of both long-term constancy and short-term variation. Simulations can also be used for the exploration of phenomena that are difficult to explore theoretically such as the lengths of runs that might be generated by tossing a coin a certain number of times, or how often two of the numbers from 1 to 6 are *missing* when a die is rolled 10 times and the outcomes recorded (over 20% of the time).

CONCLUSION

For those knowledgeable about the complexity of teaching about randomness there are, perhaps, few surprises in what has been discussed here. Knowing how to "go beyond the gimmick" is essential for teachers, if students are to develop a deep understanding of randomness in particular and probability more generally. In the non-deterministic world of chance, where phenomena can be counter-intuitive and may not always involve straightforward computational algorithms, considerable effort is required to make sure that activities are conducted effectively and over enough time to ensure that key principles are learned (the importance of committing time cannot be overemphasized: e.g., Brousseau et al. (2001) ran over 25 learning sessions for their "what's in the bag" activity). It is obviously vital that, in sharing these activities with teachers and future teachers, due emphasis is given to the principles afforded by such activities, and how to maximize their use so they are effective for student learning. It may well be helpful to be explicit about the risks and issues that have been raised here.

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