SECONDARY SCHOOL TEACHERS' KNOWLEDGE OF THE STANDARD DEVIATION CONCEPT

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In this research, we investigate teachers' statistical knowledge in relation to one measure of statistical dispersion, standard deviation. Twelve Quebec secondary school mathematics teachers were invited to respond to scenarios describing students' strategies, solutions and alternative conceptions when presented with a task based on this concept. The teachers' responses primarily helped us analyze their comprehension and practices associated with the concept of standard deviation and gain insight into how to teach this concept. Secondly, in this study we found similarities between the teachers in our study's conception of standard deviation and those commonly observed of students.

CONTEXT

At the very heart of an era when technology is becoming increasingly prevalent and with information coming from all sides, the use of statistical data is growing rapidly (Connor, Davies & Holmes, 2006). Nowadays, modern citizens need analytical skills to develop critical judgment and a personal assessment of the data encountered daily. They need to interpret results more than generate statistics for the statistical data and graphs they are continually faced with (Gattuso, 2011). Consequently, citizens must achieve statistical literacy. The role of statistics in modern society requires some pondering on teaching this discipline to train the so-called citizens of tomorrow. If the intention is to promote the development of statistical thinking among students as future citizens, then time must be devoted for developing their basic understanding to interpret statistical data. It is necessary to identify what teachers know on this subject since they help students and organize their learning activities while providing a student-friendly environment conducive to learning. We assume that concept knowledge, related to a specific concept, helps teachers to not only plan their teaching better but also to organize and manage students' learning activities in the classroom so that they may be exposed to the elements of a specific mathematical knowledge.

THE CONCEPT OF DISPERSION

A distribution's data shows variation, and although information on an important distribution aspect is provided by measures of central tendency, used alone, they may induce an incomplete representation of the distribution's reality. Hence, the need to pay attention to the variability of the statistical variable's values, which can mainly be assessed with dispersion measures demonstrating the variation of a distribution's data.

A measure of dispersion allows a data set to be described of a specific variable by providing an indication of the variability of the values within the data set (Dodge, 1993). A widely used dispersion measure for describing a distribution's variability is undoubtedly the range. It is used not only because it is easy to calculate, by simply obtaining the difference between the largest and the smallest value of a distribution, but also because the results are easy to interpret (the size of the smallest interval which contains all the data). Used alone, the range represents a limited way to measure variability since this measure of dispersion does not consider the influence that the frequency of the statistical variable's values has on variability, which explains the interest for the mean and standard deviation, two measures that indicate the variability of a statistical variable's values while this time taking into consideration all the distribution's data and allowing to find out the data spread from the distribution's center, in other words, the distribution's average. Interpreting these measures, however, seems to be a bigger challenge.

According to Delmas and Liu (2005), understanding standard deviation involves several statistical concepts such as the arithmetic mean, deviations from the mean and the relative data density around the mean, which probably explains why teachers find it difficult to teach the standard deviation concept. Proulx (2017) further highlights that viewing the mean as a measure of central tendency implies an important relationship with measures of dispersion, for analyzing data dispersion within the entire distribution, to ensure that the mean is representative of the data, that it is a valid measurement of its distribution, of its central tendency. The mean is in fact truly interesting when interpreted according to the distribution's data and its spread. Accordingly, since the standard deviation is a statistical measure linked to the mean, its real interest lies in the distribution and its central tendency, which allows, among other things, to consider the impact of outliers on this statistical measure. Let's use the example of an asymmetrical distribution where the mean doesn't truly reflect the distribution's central tendency, the median, rather than the mean, would better represent the distribution and its central tendency (the median would be linked to the interquartile range and not to the standard deviation). In short, understanding standard deviation necessitates a dynamic conception of the distribution which oversees all these concepts (Peters, 2014).

With these challenges, several studies show that understanding these statistical measures is often limited to algorithm calculation, underlining the students' difficulties in measuring variability in terms of data proximity to the distribution's central point (Cooper & Shore, 2010; Dabos, 2011; delMas & Liu, 2005; Makar & Confrey, 2005; Meletiou-Mavrotheris & Lee, 2005). This limited comprehension of the statistical measures is mostly observed when using graphical representations due to the apparent challenge of linking informal notions of variability, based on graphical displays, with more formal measures of variability. According to Garfield and Ben-Zvi (2005), being able to recognize and understand how the concept of variability appears in different graphs, especially in histograms and bar charts, is an important aspect for developing this concept when we consider that graphs' appearance is an obstacle that may induce alternative conceptions. Interpreting graphs seems to be overlooked in secondary school curriculum with the focus instead being on calculation and rules. We may take in the general look of a graph, the maximum, minimum or extreme data and still not truly understand the existing relationship between the different statistical concepts, mainly the distribution's center and the data distribution around it, leading to the concepts of average and standard deviations and to the study of variability.

Other aspects related to the variability of a statistical variable's values may certainly be documented. However, the descriptive statistics approach to measure variability in terms of data proximity to the center of the distribution using the standard deviation and its graphical interpretation, is a preferred element for building the tasks given to the teacher.

Professional Knowledge: Knowledge Linked to Practice

Nowadays, it is acknowledged that in their practice, teachers mobilize specific forms of knowledge, different than the standard forms learned in university mathematics classes (Ball & *al.*, 2008; Moreira & David, 2005, 2008; Margolinas & *al.*, 2005). Recent developments on teachers' mathematical knowledge show that some of the teachers' knowledge comes from teaching practice and is therefore linked to events arising from the teaching-learning context. (Bednarz & Proulx, 2009; Davis & Simmt, 2006, Margolinas, 2014). The above reflects the purpose of this study which is aligned with the conceptualization of professional mathematics, drawing on the works of Proulx and Bednarz (2011) and Moreira and David (2005), who consider academic mathematics and school mathematics as two distinct fields of knowledge.

For example, when teaching mathematical concepts, many mathematical events arise and are taken into account by the teacher such as reasoning (appropriate or not) that allows to give meaning to the concepts; conceptions, difficulties and errors with worked on concepts and their comprehension; various strategies and approaches to solve a problem; various representations, symbols/writings (standardized or not) to express solutions; new questions and paths to explore, etc. The above mathematical events not only refer to current concepts found in curricular documents that dictate what must be taught, but also refer to mathematical elements surrounding mathematics teaching and learning and which the teacher must use in class. The teacher's professional mathematics knowledge refers to a body of knowledge and mathematical practices structured around teaching/learning mathematics issues (Bednarz & Proulx, 2010).

Considering the above, we can identify two main dimensions related to the conceptualization of the mathematical knowledge teachers use in their practice. First, the teachers' mathematical knowledge is situated (Lave, 1988), it evolves in a specific teaching context. It is not independent of students' learning. Furthermore, the teachers' mathematical knowledge is what Mason and Spence (1999) identify as "knowing-to-act in the moment." Therefore, the teachers' mathematical knowledge builds and adapts itself in real time in reaction to the actual situation. Here we are addressing knowledge used on-the-spot, linked to the intervention used in response to an event (a script that wasn't planned, a student's question, an unexpected answer, and unplanned error, etc.).

This mathematical orientation based on practice (Even, 1993; Even & Tirosh, 1995) is at the heart of the present research. Here, secondary school mathematics teachers' professional knowledge is studied in two ways based on tasks involving statistical academic content and related students' reasoning. The first way refers to the teachers' knowledge of the standard deviation concept. Are the teachers able to perform the task and identify the students' alternative conceptions? The second way is their ability to intervene with students to help them reason from their errors.

METHOD

This exploratory project, which was conducted in French, is part of a larger research program focused on issues associated with teaching statistics with the objective of developing and analyzing training to improve the statistical experience. To address the research question, which is to find out more about secondary school mathematics teachers' professional statistical knowledge of the standard deviation, interviews organized around task resolution and previously prepared questions were developed as a data collection method to obtain participants' responses to better understand their capacity to teach this concept. The tasks consisted of analyzing teaching contents and reflecting on their integration by learners through solutions analysis and students reasoning while proposing possible interventions to develop their mathematical reasoning and comprehension. These interviews were conducted with 12 secondary school mathematics teachers in Quebec. All 12 participants were from different schools and were in the study on a voluntary basis. They had all studied statistics during their secondary school teacher training university program and they all had taught secondary school mathematics for at least 5 years. These conditions helped learn more about teachers' professional knowledge, which is directly related to mathematics teacher-training issues and to classroom practices regarding standard deviation.

The following uses a task as an example. This one was built using statistical contents analyses related to the concept of standard deviation (didactic, conceptual, and epistemological analysis; Brousseau, 1998) and inspired by analyses performed in this field (Cooper & Shore, 2008, 2010; Dabos, 2011; delMas & Liu, 2005; Meletiou-Mavrotheris & Lee, 2005).

Case example (Adapted from Meletiou-Mavrotheris and Lee, 2005)

- Step 1: Problem solving

Throughout the year, a teacher collected statistics on the quantity of water drank by Secondary 4 (grade 10) students from her school. The school has three groups of 27 Secondary 4 students. The statistics she collected are in Figure 1 here below. When looking at groups A, B and C, which distribution has the greatest standard deviation? Which distribution has the smallest standard deviation? Explain your choice.

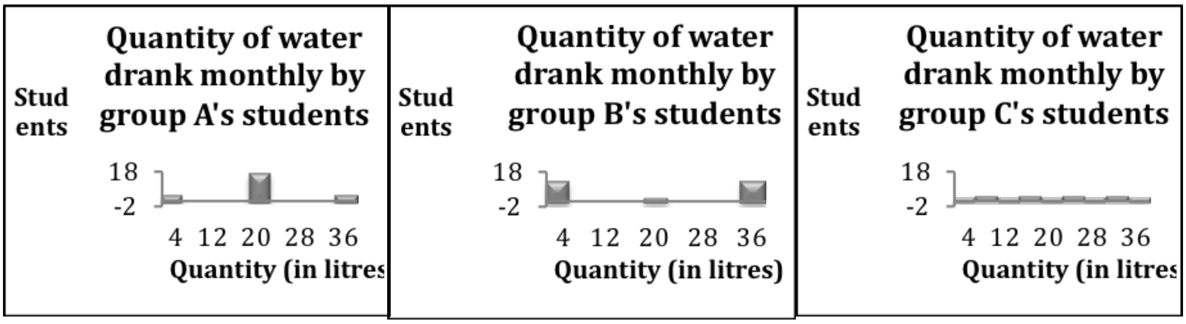

FIGURE 1—Quantity of water drunk monthly by secondary 4 students

- Step 2: Students' responses to the problem and interventions

For this question, two students came to different conclusions regarding group C. The first one said that group C's graphical representation shows the distribution with the largest standard deviation since it has the highest number of bars. This is an indicator of a large variety in the quantity of water drunk monthly by students. He then concluded that this distribution has the largest standard deviation. As for the second student, he said that the graphical representation of group C shows the distribution with the smallest standard deviation. He based his reasoning on the fact that the bars from group C's graphical representation are at an even height and therefore this distribution has the smallest standard deviation. Who is right? How will you respond to these students?

In this case example, the concept of standard deviation intervenes through the data dispersion of both distributions represented by histograms. The correct answer is: Group A has the distribution with the smallest standard deviation and Group B has the greatest standard deviation. Again, the teacher is given a task based on the students' alternative conceptions. In this question, the choice for both students' reasoning is also based on the works of Cooper and Shore (2008), delMas and Liu (2005) and Meletiou-Mavrotheris and Lee (2005). The first student's response is influenced by the number of bars which is not an indicator of a large standard deviation. By wrongly associating the group with the highest number of different values for the average deviation, group C, to the distribution with the greatest standard deviation, the student excludes the deviations' size and the students associated to each. Also, with this logic, it would be difficult to identify the group with the smallest standard deviation because the other two groups (A and B) have the same number of bars. The second student is influenced by the non-variation in the height of bars in group C's distribution. By thinking this way, the student refers to the variation of students instead of the variation in the quantity of water drunk by students monthly.

The objective of this task was to see if questioned teachers were able to assess a distribution's data spread from its center, and to observe how they could intervene on students' conceptions related to the study of standard deviation in histograms.

RESULTS

Interpretation of the students' solutions and an intervention to make with students.

Nine out of twelve teachers were not able to identify the issue or at least to see the student's alternative conceptions. It was difficult for them to intervene as they initially couldn't solve the problem and several among them shared the student's alternative conceptions in their own resolution.

The other three teachers disagreed with the reasoning of the two students and saw the issue of interpreting the data dispersion of the graphical representations in terms of data proximity to the center of the distribution. These teachers suggested an intervention that would have the students realize that group C's graphical representation does not show the distribution with the greatest standard deviation nor with the smallest. For instance, one of them explained this by referring to standard deviation.

> I wouldn't know how to show him without the calculation [the subject refers to the standard deviation's measure for the three distributions]. There are obviously other methods, but I would be quite afraid to show another way, then in another situation where it wouldn't be done in such a way they would try to do it by reasoning and make mistakes. As a teacher, I often prefer to show them the so-called "safe" methods. (Translated from French)

The other two teachers suggested explanations by referring to the data concentration around the average. For example:

> To a student who says: "I think that Group C has the smallest dispersion because the bars all have the same height." I would answer negatively because extremes are still the same, i.e. between 0 and 4 and between 32 and 36, in graph A, there is a more even dispersion around the average than in graph B where there is a cluster of people at the beginning, a cluster of people at the end and almost nobody in between. But with Group C, people are spread evenly. (Translated from French)

DISCUSSION

We join Makar and Confrey (2005) in noticing that teaching the concept of standard deviation represents an important conceptual challenge for teachers. Most teachers' interventions first show some conceptual limitations. For example, one based his thinking on the data concentration of the average class to solve the problem. By counting the number of people outside of the average class, we can identify the distribution of groups B and C as those with the smallest standard deviation, since they have the same amount of data outside of the average class. This reasoning prevents the teacher from considering the value of deviations from the mean for data outside of the average class. Teachers' interventions also highlight conceptions on statistical content already observed in pupils and university students when presented with an exercise where they must interpret the standard deviation in histograms. Most teachers were indeed influenced by the aspects associated with the distribution's shape such as the bars variation in height, the number of bars and the distribution's symmetry. These findings on the understanding of measures of dispersion echo the works of Dabos (2011) and underlines the growing interest for training teachers to understand the statistical concept they teach. They didn't notice the issue of interpreting data dispersion from the distribution's center and showed limited knowledge of this statistic measure limited to its calculation algorithm as is often the case for the mean (Watson, 2007). Let's recall the intervention proposed by one of the teachers, who without knowing what else to do, summed up the explanations to the student by referring to the standard deviation calculation.

To develop a deep knowledge of statistical measures in students, secondary school teachers must not only link the measures of central tendency to the measures of dispersion, but they must also focus on the necessary conceptual steps between learning these measures and their interpretation of graphical representation which are somewhat problematic as they contribute to misconceptions. This would make it possible to further improve the statistical learning experience offered to students.

These considerations yield results for future teacher training. Keeping in mind how challenging it is for teachers to address standard deviation, it is essential to continue reflecting on ways to contribute to teachers' professional development, mainly by gaining better understanding of the issues they encounter within this context. Of course, the present task initiates a reflection that may be used in the classroom and for future teacher training. More importantly, the realization that not all teachers were able respond quickly to students' answers and reasoning with appropriate interventions demonstrates a lack of professional knowledge of the standard deviation concept. This context raises several concerns and indicates the need to further develop statistical training for teachers enabling them to keep expanding their intervention skills to use in statistical contexts in the classroom to help develop statistical thinking in students. Research on students learning is obviously necessary as a base for creating learning situations in a teacher training context, so teachers gain better understanding on how students reason within a statistical context and think of ways to intervene to improve their mathematical reasoning and comprehension.

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