EXPLORING REALISTIC BAYESIAN MODELING SITUATIONS

Per Nilsson[1], Per Blomberg[2] and Jonas Bergman Ärlebäck[3]
[1]Örebro University, Sweden
[2]Linnaeus University, Sweden
[3]Linköping University, Sweden
per.nilsson@oru.se

The study reported in the present paper is part of a larger project, which aims to explore possibilities and challenges in developing a teaching practice that supports students' ability to model random dependent situations by a Bayesian approach. A central premise is that modeling should be based on situations that appear realistic to the students. Given this premise, the specific purpose of the present study is to identify and characterize uncertain situations that are realistic and suitable for a Bayesian treatment. The study involves reviewing some of the literature related to Bayesian applications. Based on that review we distinguish detecting (test) situations and construction composition situations as two general types of Bayesian modeling situations.

INTRODUCTION

Models are essential to mathematics. We develop mathematical models to understand the world and to predict the behavior of phenomena we encounter in the world (Blum, Galbraith, & Niss, 2007). Roughly, we can classify mathematical models into deterministic and stochastic models. Deterministic models include a number of elements and relations that completely determines a system, i.e. we can make certain predictions of how a system behaves. Stochastic models include elements that make it impossible for us to predict the behavior of the system with certainty. There is an uncertainty in how the results of a stochastic system will occur, an uncertainty we cannot trace to causal factors. Many phenomena are not suitable to deal with by a deterministic model. We need to develop models that take into account the random behavior of phenomena.

A frequentist modeling approach is based on the assumption that we can repeat a random experiment a large number of times under exactly the same conditions each time. However, in practice it is often impossible to repeat an experiment a very large number of times and to achieve exactly the same conditions in each trial. Think, for instance, of the simple experiment of throwing one die. Is it possible to throw the die from exactly the same angle and height each time? We guess not! Moreover, many situations involve the assessment of a probability when we only have data from a single, or a short termed, sample. Consider, for example, the situation that you doubt whether you turned off the coffee maker before leaving your house this morning. Although there may be frequency information on how common it is that people forget to turn their coffee maker off before leaving their house, this general frequency offers limited information about the probability that you would have done so, exactly this morning. A way to handle these situations, where we cannot meet the objective requirements of the frequentist interpretation, is to apply to the situation a probability model that is subjective in nature. A subjective model is relative to the information available and specified by the modeler of the random situation in question (Goldstein, 2006). The main objection raised against a subjective interpretation of probability concerns the scientific status of results, which is based on and varies with the observer and the information available (Batanero, Henry, & Parzysz, 2005). To meet the objections we need to organize and formalize the modeling in a scientific way. The Bayesian rule offers a way of structuring and strengthening a subjective rationality of probability modeling (Goldstein, 2006).

The current paper constitutes the initial step of a larger project, which aims at exploring and developing a framework for structuring teaching and learning of Bayesian modeling in school. A modeling sequence takes departure from some concrete, realistic situation that begs to be modeled (Freudenthal, 1983). This precondition means that we should have a rather good picture of what kind of situations that are to be modeled by a Bayesian approach. On account of that purpose, the particular aim of the present study is to identify and distinguish from a literature review various situations typical for Bayesian modeling. The analysis is motivated and guided by the idea of the

didactical phenomenology of mathematical structures, which, in turn, provides the foundations of the theory of Realistic Mathematics Education (RME) (Freudenthal, 1983).

REALISTIC SITUATIONS AND RE-INVENTION OF MATHEMATICAL MODELS
Our phenomenology analysis of Bayesian modeling will in particular be guided by two of the central, didactical components of RME: (i) the role of *realistic* situations and (ii) the role of *models* and, particularly, of *re-inventing mathematical models*.

The meaning and role of realistic situations should be considered as a critical response to a teaching tradition in which mathematics is conceived of as a list of concepts and calculation techniques, where reality and materials are used only to concretize and visualize mathematical concepts and structures. To Freudenthal, mathematical teaching should be conducted in a reversed order. It should start from realistic situations, that is, "from those phenomena that beg to be organized and from that starting point teach the learner to manipulate these means of organization" (Freudenthal, 1983, p. 32). Important to note is that realistic is not necessarily limited to real situations or phenomena. The concept should be understood from a student perspective. Something is realistic when the students find it realistic and imaginable. A computer game can thus be considered realistic. The importance of this view of realistic situations implies the need for our enterprise to invent and characterize typical phenomena to be organized by the structures of a Bayesian model.

Students should be offered opportunities to experience the advantages of organizing a realistic situation by specific mathematical structures. As the students are challenged to re-invent mathematical models on their own, "the models should 'behave' in a natural, self-evident way. They should fit with the students' informal strategies – as if they could have been invented by them – and should be easily adapted to new situations." (Heuvel-Panhuizen, 2003, p. 14). The Bayesian rule is a complex model, composed by several 'sub-models' (intrinsic structures). If a researcher or a teacher should be able to orchestrate realistic situations and discern students' informal strategies related to of some aspects of the rule, it will be necessary to de-compose the rule into a more fine-grained analytical lens of structural categories.

PHENOMENOLOGY ANALYSIS OF BAYESIAN PROBABILITY
The conditional probability of event A, given that event B has occurred, is described by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Rearranging this relation we arrive at

$$P(A \cap B) = P(B)\, P(A\,|B).$$

Changing the conditional relationship between A and B we obtain an extended version of the formula:

$$P(A \cap B) = P(B)\, P(A\,|B) = P(A)\, P(B\,|A).$$

Rearranging the right hand side, we arrive at Bayes rule:

$$P(A|B) = \frac{P(B|A)\, P(A)}{P(B)}.$$

The main feature of Bayesian modeling is that the modeler tries to estimate or up-date the probability of a parameter of a phenomenon after receiving data or evidence of how the parameter behaves (Lee, 1989). We use Θ for the set of parameters and E for evidence and arrive at:

$$P(\Theta\,|E) = \frac{P(E\,|\Theta) P(\Theta)}{P(E)} \tag{1}$$

The left hand side is the posterior probability of a parameter Θ. The posterior probability is proportional to $P(E|\Theta)$, which is the likelihood function, and involves as a factor the prior probability model of the parameter in question. Generally two parameters may be considered, which metaphorically can be conceptualized in terms of failure or success. In the case of the coffee maker we ask whether we turned the device off or not (success or failure).

Realistic Bayesian Modeling Situations

Our survey of realistic situations for Bayesian modeling is based on a brief review of the articles published between 2006-2013 in the journal Bayesian Analysis and of tasks used in mathematics education for studying aspects of Bayesian reasoning.

The right hand side of (1) builds on a two-step random structure. First, there is an uncertainty involved in the appearance of the parameter $P(\Theta)$. Then, there is an uncertainty involved in the generation of evidence $P(E|\Theta)$. In relation to these two random structures, we particularly distinguish two general types of situations from our literature review that ask for a Bayesian treatment. These situations are *detecting situations* (DS) (or test situations) and *construction composition situations* (CCS).

The Mammography problem used by Eddy (1982) represents a DS. DS are also common in situations when a system, such as a computer program or a physical system, is tested (Goldstein, 2006). Based on (expertise) knowledge about a computer program, the tester models a prior distribution of how likely it is that the program works properly (succeed) or not (fail). Next the tester develops a test instrument in order to detect how well the computer program works. However, there are many possible ways of testing a computer program. It is also nearly impossible to develop a completely certain test. Hence, the random process building up the likelihood function concerns how well the test instrument is perceived to assess the computer program. In other words, the situation does not only ask about uncertainty in terms of failure and success of the computer program. It also concerns an uncertainty in the test instrument namely, how well the test is considered to succeed in detecting a defected program or to fail in detecting an actual defect.

A CCS differs from a DS in that it does not involve the modeling of a test instrument. CCS's are comparative in nature. They beg for an inference about how a construction (i.e. a sample space) is composed or for deciding on one composition in favor of another. Say that there are two factories of the same company, producing the same kind of light bulb. The testing is not an issue of uncertainty. We simply put electrical power to the light bulb, and if it lights up it works. The prior probabilities $P(FA)$ and $P(FB)$ concern how certain we are that a light bulb are produced by Factory A and Factory B respectively, before receiving new data. In this particular situation the two prior probabilities are estimated subjectively, but defensibly, on the basis that we find it most convenient to assign the two probabilities as the proportions of light bulbs produced by the two factories. The likelihood function then concerns the composition of each factory, in terms of their ratio of producing defected light bulbs.

Let D_i define the event of i defective items. From statistical evidence and expertise knowledge of the production chain, we say that we arrive at the models: $P(FA)=0.75$, $P(FB)=0.25$, $P(D_1|FA)=0.05$, $P(D_1|FB)=0.03$. A customer has encountered a defective light bulb and we ask; what is the probability that it was produced by factory A? The Bayesian rule offers a solution

$$P(FA \mid D_1) = \frac{P(D_1|FA)P(FA)}{P(D_1|FA)P(FA) + P(D_1|FB)P(FB)} = \frac{0.05 \cdot 0.75}{0.05 \cdot 0.75 + 0.03 \cdot 0.25} \approx 0.83$$

So, before the customer returns with the defective light bulb we are certain to a degree of 75% that the bulb we gave him came from factory A. After the evidence from the customer we increase our belief of factory A to 83%. After performing the similar calculations for $P(FB|D)$, we can update our belief in factory B and compare that to our belief in factory A. This information can then be valuable for actions to improve the chain of production, within and between the two factories.

DISCUSSION AND IMPLICATIONS

Mathematical modeling has increasingly been put forward as important for our students to learn in order to cope with and be productive in their everyday and professional lives (Blum et al., 2007). In the case of probability and statistics, teaching mainly concerns the objective ideas of theoretical and frequentist models of stochastics. We know little about the possibilities and challenges in implementing a more subjective approach to stochastics. In particular, we claim there are needs for more systematic knowledge of the teaching of Bayesian modeling.

The students are not explicitly present in the current study. However, implicitly they are, as the study is motivated by the importance of basing the learning in mathematics on realistic and meaningful situations. Through a literature review, we seek to identify and characterize realistic applications of Bayesian modeling. From the review, we notice how Bayesian modeling often is applied to situations where a product or a process is to be tested and the test involves issues of uncertainty. We define these situations *detecting situations*. Another case of Bayesian modeling identified is about making inferences of the composition of a construction or a system, which is random in nature. These situations are labeled *construction composition situations*.

Implementing subjective stochastics in mathematics teaching is not unproblematic. Often subjectivity is considered to represent the lowest understanding of probability (Jones, Langrall, Thornton, & Mogill, 1997). This judgment is due to how peoples' personal assessments of a random situation often conflict with the rationality of a formal treatment of the situation. However, Bayesian modeling does not imply that every personal opinion is acceptable. It is important that teaching develop norms for what should be accepted as an argument for the modeling of the prior distribution and the likelihood function. Of course, both theoretical and frequentist arguments should be acknowledged. However, exemplified by the practices of a doctor or a criminologist, we need to develop students' ability to take into account information and expert knowledge from other sources of information as well. A Bayesian perspective does not disqualify or ignore students' personal knowledge. Instead, students' experiences and informal reasoning (cf. Makar & Rubin, 2009) are taken seriously as starting points in developing students' modeling capacity for understanding and making predictions of random phenomena.

REFERENCES

Batanero, C., Henry, M., & Parzysz, B. (2005). The nature of chance and probability. In G. A. Jones (Ed.), *Exploring probability in school: Challenges for teaching and learning* (pp. 15-37). New York: Springer.

Blum, W., Galbraith, P., & Niss, M. (2007). Introduction. In W. Blum, P. Galbraith, H.-W. Henn & M. Niss (Eds.), *Modelling and applications in mathematics education. The 14th ICMI study* (pp. 3–32). New York: Springer.

Eddy, D. M. (1982). Probabilistic reasoning in clinical medicine: Problems and opportunities. In D. Kahneman, Slovic, P., & Tversky, A. (Ed.), *Judgment under uncertainty: Heuristics and biases* (pp. 249–267). Cambridge, England: Cambridge University Press.

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Dordrecht, The Netherlands: Reidel.

Goldstein, M. (2006). Subjective Bayesian analysis: Principles and practice. *Bayesian Analysis, 1*(3), 403-420.

Heuvel-Panhuizen, M. V. D. (2003). The didactical use of models in realistic mathematics education: An example from a longitudinal trajectory on percentage. *Educational Studies in Mathematics, 54*(1), 9.

Jones, G. A., Langrall, C. W., Thornton, C. A., & Mogill, A. T. (1997). A framework for assessing and nurturing young children's thinking in probability. *Educational Studies in Mathematics, 32*(2), 101-125.

Lee, P., M. (1989). *Bayesian Statistics: An introduction*. Oxford: Oxford Universitey Press.

Makar, K., & Rubin, A. (2009). A framework for thinking about informal statistical inference. *Statistics Education Research Journal, 8*(1), 82-105.