ANALYSIS OF TEACHERS' UNDERSTANDING OF VARIATION IN THE *DOT-BOXPLOT* CONTEXT

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This study examined how dotplots and boxplots helped 23 secondary school mathematics teachers engaged in didactic activity to develop reasoning about variation. Teachers described height and head circumference by using data arranged in a dotplot and in a dot-boxplot. Their reasoning about variation was explored further with these dotplots and dot-boxplot, with a task displaying data from samples in simple boxplots, and with a task displaying data in a stacked boxplot. Teachers used mode, minimum and maximum values, and central intervals of values to describe the distributions and used nonstandard terminology such as clustering, spread out, majority, and trend in their reasoning after using dotplots. Teachers struggled to represent quartiles on a dotplot, but their reasoning about variation improved by using boxplots.

INTRODUCTION

According to Watson and Kelly (2002, p. 1), "variation is at the heart of all statistical investigation, because if there were not variation in data sets, there would be no need for statistics."

An intuitive notion of variation—for instance, understanding that data (observations) can vary, an informal aspect of the concept of variation (Garfield & Ben-Zvi, 2008)—needs to be replaced by more formal aspects of variation during a scholar's life. Formal aspects of variation are based on using measures such as range, interquartile range (IQR), and standard deviation (Garfield & Ben-Zvi, 2008) to reason about data. Developing students' reasoning about these formal aspects of variation is one of the aims of high school, college, and undergraduate statistics courses.

In some studies, participants used the range, in addition to other concepts such as maximum and minimum values and mode, to explain variation (Ben-Zvi, 2004; Silva & Coutinho, 2008; Reading, 2004; Watson & Kelly, 2002). Lehrer, Kim, and Schauble (2007) report that their participants used unconventional strategies and mean absolute deviation to describe measurement error. Other researchers noted participants' frequent use of unconventional terms such as *clustered* and *spread out* to reason about variation (Bakker, 2004; Makar & Confrey, 2005).

By using didactic activity, this study aims to verify the improvements in variation reasoning when mathematics teachers solve problems supported by sample data presented in a dotplot and in a *dot-boxplot*. The question is: can the *dot-boxplot* help mathematics teachers improve their reasoning about variation from intuitive notions to more formal notions?

METHOD

Participants

Study participants were 23 secondary school mathematics teachers (12 of them male) who were attending a Mathematics Education Masters Program. Their mean age was 36.1 years (standard deviation equal to 8), and they taught a mean of 13.2 years (standard deviation equal to 7). When asked about their knowledge about concepts of variation and graphical displays, eight reported knowing the range concept (seven of them had taught it); thirteen claimed knowledge of standard deviation (eight had taught it); and six reported knowing the dotplot (one had taught it). Twelve teachers said that the boxplot was familiar, but only five said that they knew how to interpret it. Of those five teachers, only three evidenced suitable interpretations. None of the teachers taught quartiles or the boxplot.

Didactic Activities

After reading a short text on the mathematical proportions of the human body established by Leonardo da Vinci, teachers received height and head circumference data arranged in tabular

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form for a sample of size 50. They were asked to construct a dotplot by drawing a number line on a plank of foam core and using pins to plot data. Teachers were asked to describe height and head circumference (Tasks 1 and 2) by using only the dotplots.

After completing the first and second tasks, teachers received guidelines for constructing a boxplot over the dotplot of height data from a size-50 sample (see Figure 4), what we are naming a *dot-boxplot*. The *dot-boxplot* is similar to the hat plot of Lehrer, Kim and Schauble (2007).

First, teachers were asked to identify the pin(s) that divide(s) the graph into two parts (in which each part must have the same number of pins) and to demarcate a perpendicular to the axis at this point. Next we asked teachers to look at just the first (and second) half, identify the pin(s) that divide(s) this first half (same to second half) into two parts (in which each part must have the same number of pins) and to demarcate a perpendicular to the axis at this point.

After constructing the *dot-boxplot*, teachers were asked to: a) count the number of pins in each quarter of the graph (in each part); b) calculate the percentage in each quarter; and c) choose the quarters with the highest and lowest population densities. Our hypothesis was that by using terminology such as "quarter" [one quarter (fraction) or 25% (percentage)] and "population density" [a number of pins per area]—concepts familiar to mathematics teachers—we could help them to understand quartiles and to improve their reasoning about variation by using IQR to describe data. This task ended with the request: "describe the height of people by using the new graph" (Task 3).

The fourth set of tasks consisted of describing data arranged in a boxplot for the number of hours a sample of 50 students would like to sleep during the night (Figure 1), and describing the data arranged in a boxplot for the number of pets owned by a sample of 51 people (Figure 2).

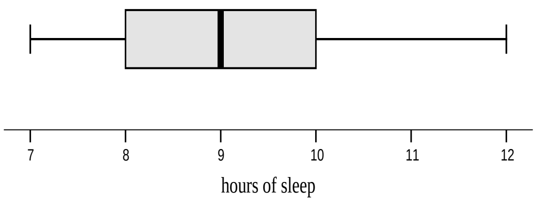

Figure 1. The first boxplot of the fourth task

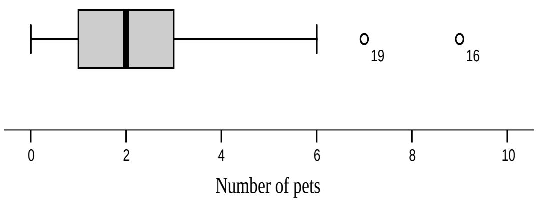

Figure 2. The second boxplot of the fourth task

The didactic intervention by the researcher was implemented after the fourth set of tasks. The second author solved the set of four tasks described above and discussed the misconceptions presented by teachers. One focus of the solutions was on measures of central tendency: computing arithmetic mean, median, and mode and comparing measures. A second focus was on measures of variation: range, IQR and standard deviation. Discussion allowed teachers to understand and compare different measures of variation for each task. Also discussed were some different ways to compute quartiles (Langford, 2006).

The last set of tasks consisted of comparing the recovery times of 60 patients who underwent three different surgical techniques (Magalhães & Lima, 2010) using a stacked boxplot, presented in Figure 3. By looking at the boxplots, teachers were asked to approximate the median and IQR of each technique and compare recovery times for the three techniques.

Teachers solved tasks in pairs, with each teacher pair identified by a letter in this paper (A, B,.., K). This didactic intervention occurred during two days in August 2013 for a total of 16 hours and was conducted by the second author. All activity was recorded in both audio and video.

Data Analysis

The four hierarchical levels of the SOLO taxonomy (Structure of the Observed Learning Outcome) of Biggs and Collis (1991, p. 65) were used to analyze teachers' answers: the prestructural level (code 0), in which "the learner is distracted or misled by an irrelevant aspect belonging to a previous stage"; the unistructural level (code 1), in which "the learner focuses on the relevant domain, and picks up one aspect to work"; the multistructural level (code 2), in which "the learner picks up more and more relevant or correct features, but does not integrate them"; and the relational level (code 3), in which "the learner integrates the parts with each other, so that the whole has a coherent structure and meaning."

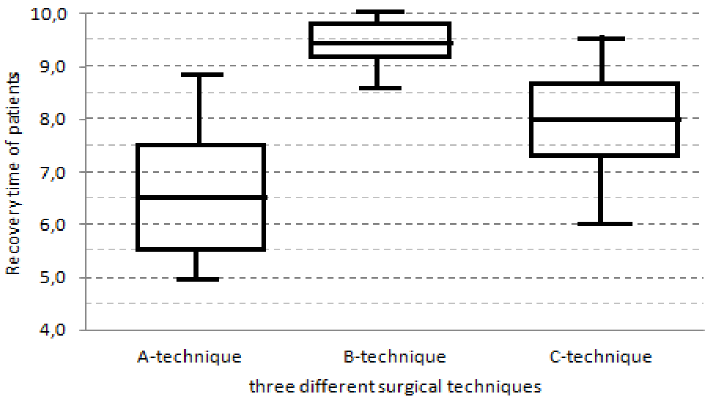

Figure 3 – Recovery times of 60 patients – fifth task

RESULTS

The majority of teachers' descriptions of height and head circumference using a dotplot were classified on the prestructural level (Table 1), because they consisted of central measures (mainly mode) without considering variation. Descriptions classified at the unistructural and multistructural level included maximum and minimum values, explanation about the distribution (*clustered* or *spread out*), and central range, named by Konold et al. (2002) a *modal clump.* The strategy of using central range was used by Pair C: *"There is more concentration between values 164 and 168 cm."* and is a very important informal aspect of variation that has appeared in papers by Makar and Confrey (2005), Silva and Coutinho (2008) and Silva, Cazorla, Kataoka and Magina (2010). However, in this study, we wanted to notice if using *dot-boxplot* would improve teachers' descriptions of variation from a central range to the formal IQR measure.

Table 1. Coding categories of responses to tasks 1 and 2 by 11 pairs of teachers

Code	Description	Height	Head
0	Incorrect responses or no acknowledgement of variation.	6	5
1	Descriptions with only one aspect of variation: "more," majority, range, maximum and minimum values.	4	4
2	Descriptions with two or more aspects of variation, without interrelating them.	1	2
3	Correct descriptions with related different aspects of variation in a formal or informal way.	--	--

In task 3, finding and representing the median and the quartiles of the distribution on a dotplot was considered a difficult task for the teachers, because of the existence of repeated values in the sample. We chose the dotplot to introduce the quartile concept as the median of each half. There were five values at the first quartile (Q1) (13^{th} position in ranking) value of 164 cm (10^{th} to 14^{th} position) (Figure 4). This fact raised two questions inherent to the concept of percentile: first, if there are repeated values, which is the exact position of the quartile? and second, if there is a value occupying the position of quartile, in which quarter of the graph (part of graph) should I count it (or them)?

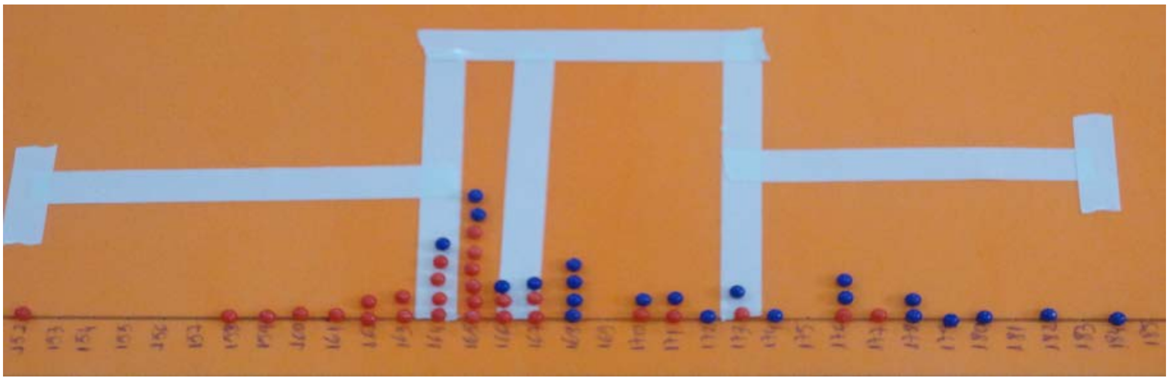

Figure 4. *Dot-boxplot* done by a pair of teachers

For the question of identifying Q1, all of the teachers identified Q1 as 164 but differed in identifying the position of Q1. Three answers were observed: a) Q1 was in the 10th position; b) Q1 was in the 14th position; and c) Q1 was in exactly the 13th position. Difficulties with repeated values are inherent in obtaining any percentile. Different responses were observed for choosing the quarters with the highest and lowest population density (Table 2). We observed answers that counted the value of Q1 in the first quarter and answers that counted it in the second quarter of the graph. (The same was the case for Q3.)

Table 2. Strategies used for obtaining the density (# heights plotted in quarter / length of quarter interval) bounded by quartiles

Density	1st quarter	2nd quarter	3rd quarter	4th quarter
Density including the height values that occupy the positions of Q1 and Q3	13/12	13/2	13/6	13/11
Density excluding the height values that occupy the positions of Q1 and Q3	12/12	12/2	12/6	12/11
Density including repeated values of Q1 in the first quarter and confusion to get Q3	14/12	11/2	14/6	11/11
Density including repeated values of Q1 in the second quarter and repeated values of Q3 in the fourth quarter	9/11	14/2	12/6	13/11

Regardless of how teachers determined the quarters for Q1 and Q3, the density was higher in the second and third quarters (Table 2), areas bounded by quartiles 1 and 3 (IQR). Only one pair of teachers failed to choose the region of higher density properly due to the difficulty of positioning the *dot-boxplot* quartiles.

At the end of the third task, teachers were again asked to describe heights for data from a sample of size 50, this time by using the *dot-boxplot*. Our hypothesis was that teachers could understand high density in relation to IQR and begin to use IQR to reason about variation (instead of using *modal clumps*), highlighting the transformation of informal reasoning to formal reasoning about variation. This improved reasoning was observed with only one pair of teachers. Two pairs of teachers chose to maintain the use of *modal clump*, such as Pair C: *"As we had pointed before, there is more concentration of height between 164 and 168 cm."* Additional reasoning strategies were observed, suggesting that participants were trying to understand IQR, sometimes based on misconceptions. For instance, one pair of teachers observed that the mean height is located in the densest area (almost always true), while another pair of teachers computed the mean for values of the highest density area (no sense).

Due to difficulties with computing quartiles and representing them in dotplots, teachers received two boxplots already constructed (Figures 1 and 2) and were asked to describe them for Task 4. Three pairs of teachers used the median value to describe the data represented in both boxplots in a mistaken way, such as Pair F (*"more concentration of students at 9 hours"*). Their answers were classified at the prestructural level. Most descriptions were classified at the unistructural level (Table 3) and included a correct understanding of IQR or other boxplot ranges but revealed misconceptions about median, as Pair D demonstrates: *"most students would like to sleep from 8 until 10 hours per night (referring correctly to IQR), but a higher percentage of students would like to sleep 9 hours (misconceptions of median)."*

The only relational answer was offered by Pair C: *"the answer is that 2! 50% of the central data ranges between one and three pets. There are two outliers: people who have 9 and 7 pets."*

Table 3. Coding categories of responses to Task 4 – describe two variables arranged in boxplots

Codes	Description	Boxplot	
		Sleep	Pets
0	Incorrect responses or no acknowledgement of variation	3	4
1	Answers with only one aspect of variation: one range, IQR, majority, percentage into some range	6	4
2	Answers with two or more aspects of variation, without fully interrelating them	2	2
3	Correct answers to requested tasks, with arguments that relate IQR, maximum and minimum values, and outliers	--	1

After the fourth task, the teaching session took place. Responses from previous tasks were discussed with a focus on measures of center, quartiles and other percentiles and measures of variation (range, IQR, and standard deviation). From teachers' reactions, we surmised that quartiles were a new concept to almost all teachers, and they were trying to accommodate new knowledge (in a Piagetian view).

At the beginning of the second day, teachers solved the fifth task. All pairs of teachers correctly read the median value for the three techniques, but one pair failed to get the IQR. Because group comparison have been shown to be an effective strategy for developing reasoning about variation (Ben-Zvi, 2004), the fifth task ended with asking teachers to compare the three techniques.

Descriptions from six pair of teachers were classified at the prestructural level because half of them chose the best surgical techniques without explaining the reasoning behind their choice and another half of them presented answers based on misconceptions about quartiles. Descriptions from five pairs of teacher were classified at the multistructural level because the descriptions consisted of important strategies such as IQR, range, concentration/spread out terms, but the teachers did not relate the ideas. For instance: "*the patients of technique A get better in five to nine days and 25% of them get better in 5 to 5.5 days. For technique B, the recovery of patients occurs in 8.5 to 10 days and 25% of them get better in 9.5 to 10 days. For C, this occurs in 6 to 9.5 days and 50% of them get better in 7 to 9.5 days*" (Pair I).

These results reveal that quartiles and boxplots were new concepts for some teachers, and they need more time to comprehend them. Additionally, the representation of quartiles in a dotplot was not easy due to repeated measures.

Another important issue to highlight is that different reasoning about variation was noticed for different types of graphs. Dotplots highlight mode (and more frequent values) and further the modal clump strategy. This strategy was not used to describe distributions in a boxplot, but some teachers developed misconceptions about median by considering it as mode. By using boxplots, teachers' reasoning about variation included IQR, total range, and Q1 to Q2 range. Finally, the *dot-boxplot* enhanced all of these strategies together, with teachers comparing different measures (center or variation) and testing their reasoning.

CONCLUSION

This research aimed to promote mathematics teachers' development of formal reasoning of variation by using IQR. Although official guidelines for secondary schools do not suggest that this concept must be taught, we supposed that learning it can help mathematics teachers grasp variation in different ways (not only standard deviation) and improve variation teaching in their classroom.

We introduced quartiles using a simple strategy as the median of each half of distribution, but it is important to remember that there are different ways of computing it (Langford, 2006). Even though we used a strategy we thought was easier than others, we need to rethink the activity. Teachers developed a misconception about quartiles and median, and they had many difficulties representing the score of quartiles in a dotplot. Results of most tasks confirm the importance of using a dotplot to notice variation, but it is necessary to think about using it to develop the understanding of quartiles.

It is important to highlight that these teachers are master-level students, were interested in learning statistics, and presented some variation reasoning that we have not noticed in other projects.

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